About Infinity, Finiteness and Finitization
(in connection with the Foundations of Mathematics)

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Part I

1. In [3] I sketched and in [4] developed to a considerable extent the Anti-traditional program of Foundations of Mathematics aimed at the banishing of beliefs from Foundations. Among the "traditional" assumptions which have been criticized and rejected as such in the new program are the assumptions of the uniqueness of "the" (intuitive) natural number series (Nn), the soundness of "mathematical induction", the objective nature (or clarity) of the notions of identity and distinction, and, finally, the assumption of the sufficiency of a language free from a formalized use of modalities, tenses, and other grammatical categories. The basic logical notions of deductions and proofs have been detached from traditional "formalization" {Footnote 1} and considerably revised; "deductions" and "proofs" in the traditional sense are referred to as "deductoids" and "demonstroids" (or as "formal" deductions and proofs). Various "prototheories" (dealing with modalities, tenses, voices, rules of indentifications and discernings, semiotical principles of using signs, notions of "relevancy", as well as of mentionings and uses) have been developed for the purposes of this program, and a "reasoning theory" - corresponding to the traditional "proof theory" - has been set up. An "ontological theory" [3-5] has been developed, essentially as a branch of the modality theory, in order to recreate without circularities large natural numbers fitting the usual demands of a theory of algorithms. The plurality of natural number series (Nn's) has been substantiated (independently of traditional "non-standard analysis" and in a more general way). A model for an "arbitrarily long" (in the new sense) fragment of a formal system $\widetilde{ZF}_k^i$ has been established and found sufficient for a consistency proof (in the revised sense) of that system. The system $\widetilde{ZF}_k^i$ is equiconsistent (in the usual sense) with the Zermelo-Fraenkel axiom system with $k$ inaccessibles ($ZF_k$). (The number $k$ was restricted in [4] to be "finite" in the sense of the ontological theory but now this restriction can be dropped or considerably relaxed). Again in the sense of the ontological theory, "finiteness" is imposed on the formal proofs in $ZF_k$ - so that this consistency proof is not expected to conflict with the second Godel Theorem because the "finiteness" in the new sense is not expressed by a traditional formula. The use of the well-known Tarski argument by means of which the consistency of $ZF_k$ is provable in $ZF_{k+1}$ suffices for a relaxation of that restriction on lengths of formal

proofs; the formal proofs may have any lengths available in the traditional meta-theories provided that their "existence" has a formal proof of length "finite" in the sense of the ontological theory. From now on, I shall adopt this meta-theoretical notion of "finiteness" without any further mentioning of the qualifying clause "in the sense of the ontological theory".

2. The proofs (in this program) are essentially of a definitional nature. That is to say, a system of definitions for all relevant notions has been developed and the uses of identifications and discernings, of acceptances of sentences, rules and aimes, and the use of modalities including those connected with aims are all imbedded in that development. Thus "proofs" in this program must be tautological (in a sense much more straightforward than the conventional use of the word "tautology" for propositional axioms).

Philosophically, there might be some limits to this program of justifications but I don't consider that a reason for not pursuing it as far as possible {Footnote 2}. That is a topic of "extreme directions" of this program [3 , pp.43-45]. In order to restrict some doubts of this nature, I have, starting in 1973-75, changed the original approach [3 ,4] to the study of the "convincingness" of a deductoid and replaced it by a new approach which can be considered as "finitistic" in a sense close to that once specified by J. Herbrand but now additionally specified in accordance with the prototheories. Now the dependence of the convincingness of the demonstroids involved in the consistency proof on the extreme directions does not seem great.

3. "Mathematical induction" (from n to n + 1) is dispensed with in the demonstroids and instead the Carnap rule ($\utilde{\text{Ca}}$) (of "infinite induction" {footnote 3}) is used. Of course, this (apparently) clashes with the general finitistic trend of this program. It is important to stress that the <u>applications</u> of that rule - to be referred to as $\utilde{\text{Ca}}$'s - must always be <u>presented</u> by a text (the $\utilde{\text{Ca}}$'s <u>presentation</u>) which is a finite object and which yields the demonstroids (or, in the general case, deductoids from some fixed hypotheses) for the $\utilde{\text{Ca}}$ premises. A $\utilde{\text{Ca}}$ presentation is always connected with a presentation of the "range" of the induction variable; this range, if infinite, shall be considered as a process which consists in following a system of rules. The $\utilde{\text{Ca}}$ presentation can be described as another system of rules for obtaining the $\utilde{\text{Ca}}$ premise for each value of the induction variable (i.e. each member of the range). This is, essentially, a specification of the constructvistic approach to the rule ($\utilde{\text{Ca}}$) except that the infinite "ranges" are

now considered in a more general way. In addition, only a few shapes of Ca presentations (and these are explicitly describable by schemata) are used in the consistency proof.

Deductoids are considered in tree form and the trees are finite. Some "tops" may be filled by the Ca presentation; immediately below them the Ca conclusions shall be placed. The deductoids of the Ca premises shall not be considered as "parts" of such a tree T: these are other tests presented by the Ca presentations at the tops of this tree. These texts shall also be considered as trees and they will be called immediately or 1-subordinated to the tree T; they in turn may have deductoids immediately subordinated to them and "2-subordinated to T", and so on. Thus a hierarchy arises. Instead of a single deductoid D - say a demonstroid of a contradiction or a formal deduction from some hypotheses HS - a "regular system of texts" (r.s.t.) shall be used which consists of the "main" member (the deductoid with the needed sentence at its root) and further members h-subordinated to the "main" member (h = 1, 2,...) and also being deductoids (from the same hypotheses, if any). Such a r.s.t. shall be called a "demonstroidal" or, respectively, "deductoidal" r.s.t.; these r.s.t.'s shall be used as the formal proofs or deductions of the root sentences of their "main" member. (In a deductoidal r.s.t., the text HS shall be common for all members - i.e., in each member, its hypotheses, if any, shall belong to HS.) For the consistency proof it will suffice to consider the r.s.t.'s in which the numbers h are restricted by a fixed number (which may depend on the "complexity" of the formulas in the "main" member). This is one of the reasons why the proof can be finitized.

4. There are other subjects with which the consistency proof must be concerned.

(a) The use of substitutions of termoids {Footnote 4} t for a variable $\bar{x}$; in each case when such a substitution is done, the 'arrow' $\bar{x} \to t$ shall "precede" the formula A(t) thus obtained from A($\bar{x}$) so that the "strong" Bernays axioms {Footnote 5} shall be written in the form

$$\bar{x} \to t \vdash \forall.\, \forall \bar{x} A(\bar{x}) \supset A(t)$$

$$\bar{x} \to t \vdash \forall.\, A(t) \supset \exists \bar{x} A(\bar{x})$$

where the variable $\bar{x}$ does not occur in t, and the termoid t, if it contains variables, contains at most one occurrence of function symbols. (These restrictions actually don't weaken the system and are aimed at the decomposition of substitutions into "simple" ones.)

(b) The use of identifications of terms for interpretational purposes as well as for establishing that it is possible to make "timely"

identificattions of any two common members of the ranges of two identified variables. "Timely" means, in particular, "previous to any use of these identifications in the whole process of developing of all relevant identifications" and that it is necessary to keep in mind that (in order to avoid unsolved prototheoretical problems) any two objects may be identified <u>only when they are arrived</u> in the process (or activity) of obtaining them. In the prototheories I describe some classes of identifications whose "fulfillability" - i.e. the possibility to present them as the classes of events of fulfillable processes - suffice for the semiotical purposes at issue;

(c) The r.s.t. members subordinated to a given one, R, arrive only provided R is arrived, and all parts of a r.s.t. member - including the appearances of termoids at the members as well as parts of the termoids - arrive together with that member.

(d) All relevant appearances of the (usual) logical postulates must be justified in full agreement with these tasks.

5. The use of arrows is subject to some rules. In particular, the instances of ($\underset{\sim}{\mathrm{C}}$a) have the shape

$$\frac{\ldots\ \bar{x} \xrightarrow{s} \underline{x},\ \Gamma^{t}(\underline{x}),\ \Delta^{u} \vdash A(\underline{x})\ \ldots}{\Gamma(\bar{x}),\ \Delta \vdash \forall \bar{x} A(\bar{x})}$$

where $\underline{x}$ is a value of the "induction variable" $\bar{x}$ and the sentence displayed between the dots - called the "<u>x-premise</u>" - has, to the left of $\vdash$, the arrows $\bar{x} \to \underline{x}$, $\Gamma(\underline{x})$, $\Delta$ each of which occur $\leq$ s, t, u times, respectively; here s, t, u $\geq$ 0 and may depend on $\underline{x}$. The arrows $\Gamma(\bar{x})$, $\Delta$ are of the shape considered above (see page 3) and the arrows $\Gamma(\bar{x})$ all contain $\bar{x}$, at the right side only, are said to "<u>absorb</u>" the arrows $\Gamma(\underline{x})$ (obtained from them by the substitution of $\underline{x}$ for $\bar{x}$; arrows in $\Delta$ don't contain $\bar{x}$ at the right sides; $\Gamma(\bar{x})$, $\Delta$ may be empty. $A(\bar{x})$ contains $\bar{x}$ freely while $\forall \bar{x} A(\bar{x})$ must be a closed formula. The arrows $\bar{x} \to \underline{x}$ are said to be "<u>swallowed</u>" by the use of ($\underset{\sim}{\mathrm{C}}$a).

($\underset{\sim}{\mathrm{C}}$a) is an "unusual" rule in that its premises, though "presented", are not supposed "present" in each application. The point is that the "$\underset{\sim}{\mathrm{C}}$a presentations" - i.e. the presentations of the applications of ($\underset{\sim}{\mathrm{C}}$a) {Footnote 6} - actually present a certain <u>method</u> $\mathfrak{M}^{A(\bar{x})}_{\bar{x} \to \underline{x}}$ of obtaining, for each value $\underline{x}$ of $\bar{x}$, a r.s.t. member for the $\underline{x}$-premise. But this member $P(\underline{x})$ (as well as the $\underline{x}$-premise) can get an occurrence only when $\underline{x}$ - as well as any termoid $\phi(\underline{x})$ to occur in $P(\underline{x})$ - is presented <u>and</u> is present and the same for any termoid $\Phi(\underline{x})$ to occur in $P(\underline{x})$. That is indicated by the dots in the figure used above for the description of the ($\underset{\sim}{\mathrm{C}}$a). (Occasionally, the arrows don't occur in the

sentences; in such cases the sentences become $\vdash F$ where $F$ is a closed formula; the sign "$\vdash$", when not "preceded" by an arrow or another formula, may be dropped.)

The rule (MP) takes the shape

$$\frac{\Gamma \vdash A \qquad \Delta \vdash A \supset B}{\Gamma, \Delta \vdash B}$$

where $\Gamma$; $\Delta$ are texts of arrows (possibly empty; see the remark above ).

This rule has a version, $(\overrightarrow{MP})$, which deals with the removal of "correct arrows":

$$\frac{\Gamma; \quad \Gamma \vdash F}{F}$$

- i.e. when all arrows $\Gamma$ in an accepted sentence $\Gamma \vdash F$ are accepted, the formula $F$ of the sentence may be accepted. This rule $(\overrightarrow{MP})$ is needed in order to obtain just the formula $F$, not $\Gamma \vdash F$, at the root of the "main" member of the r.s.t. And that shall be its only use allowed in r.s.t.'s. (That is a restriction on the r.s.t.'s at issue, and its rationale consists in that arrows must be preserved "till the very end" for purposes of the analysis of their use.) Also a generalization of $(\overrightarrow{MP})$

$$\frac{\Gamma \qquad \Gamma, \Delta \vdash F}{\Delta \vdash F}$$

- with the same restrictions on its use - may be accepted. However, I shall not use that in this work.

There shall be a further "finitary" rule of inference, the "conjunction rule" (cr):

$$\frac{\Gamma_1 \vdash A_1 \ \ldots \ \Gamma_m \vdash A_m}{\Gamma_1, \ldots, \Gamma_m \vdash \&(A_1, \ldots, A_m)} .$$

No identifications or discernings of parts of two different premises are allowed above or below the bar. (If this restriction is dropped, the rule shall be denoted by (CR); that is a derived rule, at least if $m$ is a number available by the ontological theory.)

If all $\Gamma_1; \ldots; \Gamma_m$ can each be identified with a single text $\Gamma$ or arrows, then also the "reduced" form, $(\hat{cr})$, of this rule:

$$\frac{\Gamma \vdash A_1 \ , \ldots, \ \Gamma \vdash A_m}{\Gamma \vdash \ \&(A_1 \ , \ldots \ A_m)}$$

shall be considered as applicable.

Some further rules - called the "covering rules" shall be accepted in the r.s.t.'s in order to help bring the arrows of the $\underset{\sim}{x}$-premises of $\underset{\sim}{Ca}$'s into the shape above. They shall allow one to replace "constant" arrows - i.e. arrows containing no variable in their right sides - by other constant arrows which "cover" the former; the "covering" arrows shall express stronger 'conditions' than the 'covered' ones. Say, if a number $\underline{n}$ belongs to the range of a number variable m, then $\underline{n}$-1, and any smaller number $\underline{p}$ shall also belong to the range. (That is a requirement imposed on the ranges: they shall always contain, together with an event e of a process - such as a Nn - any event d that arrives in the process earlier than e.) The covering rules allow one to replace $m \to \underline{p}$ (or $m \to \underline{n}-1$) by $m \to \underline{n}$. In this way, the function symbols $\chi(n)$ with the property $\chi(n) \leq n$ can be avoided in the "absorbing" arrows - and, ($\underset{\sim}{Ca}$) being the only source yielding "variable arrows" (i.e. arrows with variables in the termoids of their right sides), also in the variable arrows in the r.s.t. members. (A further rule shall introduce superpositions of function symbols in arrows; but surely, a function $\chi(n)$ with $\chi(n) < n$ for some n cannot be obtained by superpositions from functions $\tau$ with the property $\tau(n) \geq n$ for all n.) Also n-ary function symbols with $n > 1$ can be avoided by means of the covering rules: say, $m \to \underline{n} \cdot \underline{q}$ can be covered by $m \to [\max(\underline{n}^2, \underline{q}^2)]$ and then the latter - by two arrows $m \to \underline{n}^2$, $m \to \underline{q}^2$. By this means the variable arrow $m \to n \cdot q$ shall be avoided in derivations of the r.s.t.'s in favor of two arrows $m \to n^2$, $m \to q^2$ with only unary function symbols in the right sides.

As soon as the formal language used in the r.s.t. members in well described, it becomes possible to formulate, for each particular r.s.t. I have to deal with, the concrete shapes of the covering rules, without any use of variables for functions in them. They may have two premises, say,

$$\frac{\underline{p} \leq \underline{n} \qquad m \to \underline{p},\ \Gamma \vdash F}{m \to \underline{n},\ \Gamma \vdash F}$$

or

$$\frac{\underline{n}\text{-}1 \leq \underline{n} \qquad m \to \underline{n}\text{-}1,\ \Gamma \vdash F}{m \to \underline{n},\ \Gamma \vdash F}$$

(In the latter case, it is easy to stipulate that the premise $\underline{n}$-1 $\leq$ $\underline{n}$ may be dropped.) They may also have two or more arrows in the conclusions such as in e.g.

$$\frac{m \rightarrow \max(\underline{p},\underline{q}),\ \Gamma \vdash F}{m \rightarrow \underline{p},\ m \rightarrow \underline{q},\ \Gamma \vdash F}$$

(and similarly for $m \rightarrow \max(\underline{p},\underline{q},\underline{r})$ and so on.)

As soon as a formula is preceded by the arrows $\vec{x} \rightarrow t(\vec{y})$, $\vec{y} \rightarrow s$ (and, possible, others) and this $\vec{y}$ is to be identified, then also the "composition" $\vec{x} \rightarrow t(s)$ of the arrows has to hold (if the arrows are "correct" - i.e. each value of the right side must be a value of the left side in each of the arrows.) Let $id_{\vec{y}}$ denote this "compositional identification". The formation of the "compositional arrows" has to be continued until each of the compositional identifications (which occur in the given text $\Gamma$ of arrows), $id_{\vec{y}}$, is used. So, if a third arrow, $\vec{v} \rightarrow m(\vec{x})$, with $\vec{x}$ identified here and in $\vec{x} \rightarrow t(\vec{y})$, occurs, then the composition $\vec{v} \rightarrow m(t(s))$ of $\vec{v} \rightarrow m(\vec{x})$ and $\vec{x} \rightarrow t(s)$ has to be formed, and similarly if $s$ contains a variable $\vec{w}$ and a further arrow, $\vec{w} \rightarrow u$, occurs among the arrows preceding the formula, with this $\vec{w}$ identified in the text of these arrows; occasionally the text can contain more than one compositional identification of $\vec{y}$ - say, if $\vec{w}$ is again $\vec{y}$ and both are identified; then each compositional identification of $\vec{y}$ has to be counted separately in this procedure - and that applies to each variable ($\vec{x}$, $\vec{w}$ a.o.) having such an identification in the text.)

The compositional arrows specify the "closedness properties" of the ranges of the variables; the ontological theory provides the possibility of getting "ranges" which satisfy this property for all arrows which occur in a r.s.t. at issue. More specifically, the "closedness property" expressed by an arrow $\vec{x} \rightarrow \phi(\vec{y})$ consists in that, for any value $\underline{y}$ of $\vec{y}$, $\phi(\underline{y})$ shall be a value of this $\vec{x}$. Thus, the range of $\vec{x}$ shall be "closed" w.r.t. the applications of $\phi$ to the values of $\vec{y}$. (This terminology agrees with the ordinary use of the word "closedness" in that the values of $\vec{y}$ always shall be among those of $\vec{x}$; $\vec{x}$, $\vec{y}$ shall always refer to events of the same process, though not necessarly considered at the same stage. I wish to stress that the "closedness" in this sense does not entail that $\phi(\phi(\underline{y}))$ shall be a value of $\vec{x}$ - and, perhaps, the term "semi-closedness" is preferable.)

The order of the arrows in the texts $\Gamma$ of the sentences $\Gamma \vdash F$ shall be immaterial. No rule shall entitle one to drop repetitions of arrows (except the rule ($\hat{c}r$) above, which is to be used with caution: dropping the repetitions can result in losing some of the "closedness properties" of the ranges of the variables).

The rule of "thinning", (Th):

$$\frac{\Gamma \vdash F}{\Delta, \Gamma \vdash F}$$

shall be accepted only

(a) in order to repeat, in $\Delta$, some occurrences of the arrows in $\Gamma$ (say, if it is desirable to consider the texts $\Gamma^{t}(x)$, $\Delta^{u}$ in the $\underset{\sim}{x}$-premises of $\underset{\sim}{Ca}$'s as containing exactly t, resp. u occurrences of each arrow which belongs to is);

(b) in order to include in $\Delta$ all compositions of arrows from $\Gamma$.

For purposes different from those of the formal deductions, I am ready to admit also some further uses of (Th) provided that no complication arrises. Arrows thus appended shall be correct and not create any 'cycle' to be considered below. In particular, the following item can have some theoretical value:

(c) the arrows $\vec{x} \to \underset{\sim}{x}$ to be swallowed may be introcuced in the premises of a $\underset{\sim}{Ca}$.

For the purposes of the use of arrows, it will suffice to use a), b) only at the bottom of each r.s.t. member (before the application of $(\vec{MP})$ in the case of the "main" member; firstly a) will be used, if needed, and then b)).

The "strong Bernays Axioms" (see p. 3 ) - as well as other axioms - shall be postulated only with the void closure string $\forall$. in their formulas. This formulation **presumes** that non-closed formulas can appear in r.s.t. members only as parts of the closed ones. It follows that the arrows $\bar{x} \to t$ in them shall be constant. The variable arrows occur only as the "absorbing" ones, which are then "repeated" below the bar (they may occur in the "repeated" arrows "$\Delta$" of forthcoming $\underset{\sim}{Ca}$'s) - and, besides that, new variable arrows can occur only as compositions.

These are the rules for arrows. They must be supplied with the rules governing the use of the collations of their parts. (Say, $\bar{x}$ and respective parts of t are to be identified in the arrow and in the occurrence at the formula, in each strong Bernays axiom. $\vec{x}$ in the absorbing arrows $\Gamma(\bar{x})$ of the $\underset{\sim}{Ca}$'s are to be identified in the arrows and in the explicit quantifier; the "rewritings" below the bar of the finitary rules keep the identifications as do the steps by a), b) of Th's also, and so on {Footnote 7}.)

For each r.s.t. member, R, it suffices to consider each range as containing its "initial events" (from which the process of obtaining the range starts), closed with respect to functions which occur in this member - including those in the compositions - and, in addition, being an extension of or coinciding with each range of the variables of the

same sort which occur in the r.s.t. members $Q$ to which $R$ is h-subordinated for some $h > 0$ - but not w.r.t. superpositions or iterations of these functions {Footnote 8}.

That is, it suffices to consider the processes yielding the ranges as developing only so far as is needed in accordance with the last paragraph. The infinite processes may thus be reduced to finite sets - which are not, however, to be accomplished at the moment of their consideration. That is the second reason why the consistency proof can be finitized.

6. A text of arrows $\bar{x}^1 \rightarrow t^1(\bar{x}^2)$, $\bar{x}^2 \rightarrow t^2(\bar{x}^3)$, ..., $\bar{x}^m \rightarrow t^m(\bar{x}^1)$, where each $t^i$, if present, is a function symbol, $i = 1, \ldots, m$, is called a "C-cycle" (because $\bar{x}^1$ is edentified in its two appearances). This C-cycle is called dangerous if at least one $t^i$ is a function symbol. (This term refers only to the r.s.t.'s at issue and to the function symbols which occure in their members.)

A great deal of the system $\widetilde{ZF}_k^i$ can be developed without (dangerous) C-cycles arising. In particular, the infinity axioms (including those for inaccessibles), the mathematical induction principles and many other devices belong to that part of $\widetilde{ZF}_k^i$. The substitutions of termoids containing a variable $n$ in a formula which already contains $n$ must be used cautiously. This constitutes an interesting fragment of $\widetilde{ZF}_k^i$ - or of $\widetilde{ZF}_k$ - for which finitistic models exist, and that is important for the consistency proof for the fragment.

The word "fragment" is here applied to a "ZF-like system" ¬ such as $\widetilde{ZF}_k^i$ or $ZF_k$ - in two senses:

(a) for any number $\ell$ (finite in the revised sense) all formal proofs of length $\leq \ell$ form a fragment of the system,

(b) some restrictions are imposed on the use of the logical postulates (actually only on the use of variables in them).

I study the fragments in the sense (b) - and, for any such fragment, the fragments in the sense (a) shall be considered for any $\ell$ for which the "$\ell$-consistency" of the (restricted) system is to be established.

The establishing of a finite model actually makes up only a part of a consitency proof, not sufficient for such a proof as a whole. In view of the Ultra-intuitionstic criticism, the "existence" of a finitistic model does not grant the consistency of the respective fragments (in both senses (a) and (b)) unless a further analysis establishes the fulfilment of all requrements of the prototheories - see especially (b)-(d) on pp. 3-4 . I don't exclude automatically a possibility of a finitistic model for a contradictory fragment. Because of that, I refer to the constructions of the finitistic models without such an accompanying analysis

only as "semi-proofs of consistency" or the like.

## Part II

1. In this part I shall sketch a way of constructing finite models for fragments of logical systems dealing with infinite processes. Even an inconsistent system can have fragments with finite models. In my opinion, the existence of a finite model for a fragment of a theory does not without additional research guarantee the consistency of the fragment. I shall call a fragment with a finite model "consistent-like", or "semi-consistent" (or even "half-consistent".)

2. For a traditional thinker, semi-consistency may seem to be almost the same as consistency. But there are philosophical doubts about this and, in any case, the semi-consistency of a fragment of a formal system is not the same metatheoretical statement as its consistency. A formal system can be called "semi-consistent" if any of its finite fragments is (say, if all of its theorems provable in $\leq n$ steps is semi-consistent for any n). The semi-consistency does not necessarily imply the consistency of the **system**; if the former is expressible by a Gödel style formula, then this formula may not imply (in the system) the consistency formula. Nevertheless, the semi-consistency of a formal system is an interesting property which can be used for the consistency proofs.

3. The systems I have in mind deal with several Nn's - "natural number series" - or connected processes. These processes, $D^j$, make natural models for ZF set theory without the infinity axiom. Instead, $D^j$ has some number $\underline{z}_j + 1$ of "zeros" - $a_0^j, \ldots, a_{\underline{z}_j}^j$ - and two "leading operations": $[\underline{x}_j]$ (which can be interpreted as the "unit set" with $\underline{x}_j$ as its only member) and the "unions" $\bigcup(\mathfrak{z}_j^1, \ldots, \mathfrak{z}_j^m)$ ($m \geq 1$) of the "kernels" $\mathfrak{z}_j^c$, $c = 1, \ldots, m$, which are the zeros of $D^j$ or have the shape $[\underline{x}_j]$. At a given point, the zeros are "void" but this will change when j is taken to be a varying magnitude. Two events $\bigcup(\mathfrak{z}_j^1, \ldots, \mathfrak{z}_j^m)$, $\bigcup(\bar{\mathfrak{z}}_j^1, \ldots \bar{\mathfrak{z}}_j^n)$ shall be indentifiable if the sets $\{\mathfrak{z}_j^1, \ldots, \mathfrak{z}_j^m\}$ and $\{\bar{\mathfrak{z}}_j^1, \ldots, \bar{\mathfrak{z}}_j^n\}$ of their kernels are (in the usual way) and if $m = 1$ then $\bigcup(\mathfrak{z}_j^m)$ shall be indentifiable with $\mathfrak{z}_j^m$.

I shall run over the numbers 0, 1,..., k where k is any obtained number. Also a Nn, $D^{-1}$, shall be fixed; the number $\underline{z}_0$ shall be con-

sidered as exceeding all $D^{-1}$ numbers. There will be other Nn's, $N_j$, fixed so that $z_0$ belongs to $N_0$ and each $N_j$, $0 \leq j < k$, shall be "shorter" than some number $\dot{z}_{j+1}$; $z_{j+1}$ shall be defined as $2\dot{z}_{j+1}$ and considered as a $N_{j+1}$-number. The leading operations of $D^j$ shall be considered as applicable any $N_j$-number of times.

The Nn's $N_j$ shall be taken as closed under all operations needed in order to 1-1 enumerate $D^j$ by $N_j$-numbers. Also $D^{-1}$ shall be taken as rich enough to contain sums $\alpha + \beta$ of its numbers $\alpha, \beta$ and any number which may be obtained and specified for purposes of application.

Thus for $j = 0, \ldots, k-1$, the events $x_j$ of $D^j$ shall be assigned their $N_j$- numbers, $\nu(x_j) < \dot{z}_{j+1}$; they shall be assigned also the $N_{j+1}$-numbers $z_j + \nu(x_j)$ (to be denoted by $\nu^{j+1}(x_j)$); for any $f = j + 2, \ldots, k$, $\nu^f(x_j)$ shall be defined as the $N_f$-number "equivalent" to $\nu^{j+1}(x_j)$. The zero $a^f_{\nu^f(x_i)}$ is taken as the unit set with $x_i$ as its only member, $f = i + 1, i + 2, \ldots, k$, and $i = 0, \ldots, k - 1$. The $D^{-1}$-numbers shall be denoted by $\alpha_1$ (its zero), $\alpha_2, \alpha_3, \ldots$; the "leading operation"- the successor function in $D^{-1}$ - shall be denoted by $p(\ )$, so that $p(\alpha_i)$ shall be $\alpha_{i+1}$. For any $D^{-1}$ number $\alpha_i$ and $j = 0, \ldots, k$ the zero $a^j_i$ of $D^j$ shall be taken to be the unit set having $\alpha_i$ as its only member. A "union" $\bigcup(\mathfrak{z}^1_j, \ldots, \mathfrak{z}^m_j)$ possesses as its members of $D^j$ those and only those $x_j$ for which $x_j \varepsilon \mathfrak{z}^c_j$ holds for some $c$, $1 \leq c \leq m$, the sign "$\varepsilon$" being used to express the membership relation; for the "units set" $[x_j]$, $u_j \varepsilon [x_j]$ holds iff $u_j$ is $x_j$.

Thus the membership relation - defined originally only on each $D^j$, $j = 0, \ldots, k$, is extended. No further case when $x_i \varepsilon y_j$, $-1 \leq i, j \leq k$, holds will be assumed and unless $x_i \varepsilon y_j$ holds by the stipulations above, $\neg x_i \varepsilon y_j$ obtains. Finally, for $i \neq j$ and any $x_i, y_j$ in $D^i, D^j$, $\neg x_i = y_j$ shall be accepted $(-1 \leq i, j \leq k)$.

4. This constitutes a description of a kind of model for a version $\widetilde{ZF}_k$ of the system ZF for the set theory with an infinite set of "individuals" ("Urelemente") and at least $k$ inaccessible alephs; the extenstonality, Fundierung, and choice axioms are not included in $\widetilde{ZF}_k$. The objects of the universe for $\widetilde{ZF}_k$ are of $k + 2$ "kinds" or "sorts", $j = -1, 0, \ldots, k$ and for each $j = 0, \ldots, k$ the processes $D^i$, $i = -1, 0, \ldots, j$, with the relations $\varepsilon$ and $=$ restricted to them form a model for $\widetilde{ZF}_j$. Also for $j = 0, \ldots, k$ a function $q(x_j)$ shall be "postulated" in $\widetilde{ZF}_j$ with the axioms $[x_j \subsetneq q(x_j)]$, and $[x_j \subseteq y_j \ \& \ y_j \subseteq x_j \supset q(x_j) = q(y_j)]$. These axioms are needed for the relative consistency proof of

the extensionality axiom.

The infinity axiom (in the Dedekindian form) is implied by two "small Peano axioms" for $D^{-1}$ and the axiom $\forall\alpha(\alpha \in \bigcup(a_0^0, \dots, a_{z_0}^0))$ included in $\widetilde{ZF}_k$ (and easily "verifiable" by the model); here $\alpha$ is a variable "of the kind -1" (i.e. for the $D^{-1}$ numbers). The 1-1 mapping of $\bigcup(a_0^0, \dots, a_{z_0}^0)$ into itself is achieved by the successor function $p$ of $D^{-1}$ (and the set of pairs $\langle p(\alpha), \alpha\rangle$ is available by the rest of axioms, including the Separation axiom). For $j = 1, \dots, k$ the event $\bigcup(a_0^j, \dots, a_{z_j}^j)$ is the universe of $\widetilde{ZF}_{j-1}$. The model for $\widetilde{ZF}_{j-1}$ is "supercomplete" in the sense of [1] (because the Separation and Replacement axiom schemata hold without restriction of the variables to the "kinds" $< j$) - and that implies that its "ordinals" are "absolute" in the iniverse of $\widetilde{ZF}_k$ and make an inaccessible aleph, $On_{j-1}$ $(j = 1, \dots, k)$.

The logic of $\widetilde{ZF}_k$ is the $k + 2$ sorted (classical) predicate calculus (with equality and these "disjoint" sorts or kinds).

5. Surely this model is highly problematic from the traditional viewpoint because of the use of different Nn's $N_j$, $j = -1, 0, \dots, k$. But the possibility of such Nn's - hinted at by the well-known "non-standard models" (as well as by some intuitive examples [2]) - never was excluded by purely logical means. In another place I have shown that it can be supported by special research of a model-theoretical and semiotical nature [3-5].

The consistency proof for $\widetilde{ZF}_k$ deals only with formal proofs in $\widetilde{ZF}_k$ having "length" {Footnote 9} among numbers which admit a construction of a sort which has been described in [5]. There I developed such constructions; in any case, $n$ being constructed, $2^n$ becomes constructible and thereafter the modality theory entitles one to consider $2^n$ as constructed and thus a number "available" for use in the quality of this $n$; thus the constructions may be repeated $2^n$ times with larger and larger numbers becoming available. A number $m$ becomes "constructible" as soon as a larger number does, and a small number-like 2 or 10 - is considered as constructed ab initio. Thus some "predicative" numbers arise independently of any axiomatic system and any use of variables ranging over "all" these numbers. Only these numbers will be used for "length of proofs" in the consistency problem. That is, a number $m$ is (called) "predicative" if it is obtainable by some constructions introduced without any reference to $m$, or to a greater number, or to "numbers" in general, so that these constructions produce $m$ from numbers already obtained; any g-fold superposition of such constructions is considered as a construction available only if $g$ is obtained.

(This notion seems non-expressible in the traditional formal languages.)

Such a number $\underline{\ell}$ being fixed, and there being an alleged contradiction proof $Ctd_{\underline{\ell}}$ of length $\leq \underline{\ell}$ in $\widetilde{ZF}_k$, only finitly many $N_j$-numbers become of relevance to $Ctd_{\underline{\ell}}$, so that the rest of the $N_j$-numbers can be disregarded in the study of $Ctd_{\underline{\ell}}$. Their "existence" will not be used in the study of $Ctd_{\underline{\ell}}$, the number $\underline{z}_{j+1}$ exceeding the "non-alien" (to the study) $N_j$-numbers becomes actually available. To be sure, the Nn's $N_j$ (for $j = -1$, $N_j$ is $D^{-1}$) must be supposed "closed" w.r.t. a unary strictly monotone function $s(\ell)$; actually the addition of $2^{\ell}$ and even of $2^{\underline{\ell}}$ (or of $2^{2^{\underline{\ell}}}$) suffices (if $\underline{\ell} \geq 8$). Thus the Nn's $N_j$ become replaceable by finite "segments" of numbers.

6. For $j = 0, \ldots, k$ there is an important function, $f(\underline{x}_j)$, called the "degree" or "type" of $\underline{x}_j$. $f(\underline{x}_j)$ shall be defined thus:

$$f(a_h^j) = 0,\ h = 0, \ldots, \underline{z}_j;$$

$$f([\underline{x}_j]) = f(\underline{x}_j) + 1;$$

$$f(\bigcup(\mathfrak{z}_j^1, \ldots, \mathfrak{z}_j^m)) = \max(f(\mathfrak{z}_j^1), \ldots, f(\mathfrak{z}_j^m)).$$

The "union" $\underline{x}_j \cup \underline{y}_j$ where $\underline{x}_j$ is $\bigcup(\mathfrak{z}_j^1, \ldots, \mathfrak{z}_j^m)$ and $\underline{y}_j$ is $\bigcup(\bar{\mathfrak{z}}_j^1, \ldots, \bar{\mathfrak{z}}_j^n)$ shall be defined to be $\bigcup(\mathfrak{z}_j^1, \ldots, \mathfrak{z}_j^m, \bar{\mathfrak{z}}_j^1, \ldots, \bar{\mathfrak{z}}_j^n)$.

For any $N_j$-number $m$ an event $h(m)$ - the $m^{th}$ "layer" of $D^j$ - shall be defined to be $\bigcup\{\mathfrak{z}_j \mid f(\mathfrak{z}_j) \leq m + 1\}$ which is the "initial cup" applied to the set of all kernels $\mathfrak{z}_j$ of $D^j$ where $f(\mathfrak{z}_j) \leq m + 1$. For any event $\underline{x}_j$ of $D^j$, $hf(\underline{x}_j)$ shall be defined as $h(f(\underline{x}_j))$. (In particular the function $q(x_j)$ mentioned earlier will be taken to be $hf(x_j)$). It is easily shown that

$$(0) \qquad f(hf(\underline{x}_j)) = f(\underline{x}_j) + 1 .$$

For $j = -1$, $f(\underline{\alpha})$ is defined to be $p(\underline{\alpha})$.

7. The language of $\widetilde{ZF}_k$ shall be extended so as to include the constant and function symbols introduced so far. The form of the logic shall be so chosen that only closed formulas will be allowed as axioms or theorems. The substitutions (of termoids for variables) shall be presented by the "strong Bernays axioms"

$$(SBA) \qquad \bar{x}_j \rightarrow t_j \quad \vdash \forall\ .\ \forall \bar{x}_j\ A(\bar{x}_j) \supset A(t_j)$$

(and the dual form with $\exists \bar{x}_j$). Here $\bar{x}_j$ stands for any variable of the kind $j$ and $t_j$ for any termoid of this kind which is either "con-stant" (i.e. contains no variable) or contains only one variable distinct from $\bar{x}_j$ and in this case it contains at most one occurrence of of a function symbol. The use of unary function symbols with variables suffices for the present purposes and simplifies the exposition. In the rest, these restrictions on $t_j$ don't weaken the system (cf. (a) on p.3 above).

Suppose $t_j$ is of the form $\phi(u_j)$ where $u_j$ is a variable. Then I require that the function symbols in the formal language be choosen so that for any value $\underline{u}_j$ of $u_j$

$$(*) \qquad f(t_j) \leq f(\underline{u}_j) + 1 \quad (j = -1, 0, \ldots, k).$$

The rest of the logical postulates - besides the strong Bernays axioms - constitute a part of logic which I call the "Weak Predicate Calculus" (WPC).

Some of the non-logic axioms of $\widetilde{ZF}_k$ shall be "preceded" by arrows. For example, the power set axiom shall be given as

$$y_j \rightarrow hf(x_j),\ t_j \rightarrow z_j \vdash \forall x_j \exists y_j \forall z_j (\forall t_j (t_j \varepsilon z_j \supset t_j \varepsilon x_j) \supset z_j \varepsilon y_j) \qquad (j = 0, \ldots, k)$$

and the axioms

$$(\#) \qquad \forall x_j \forall z_j (x_j \subseteq z_j \ \&\ z_j \subseteq x_j \supset hf(x_j) = hf(z_j))$$

shall be "preceded" by the arrows $t_j \rightarrow z_j$, $t_j \rightarrow x_j$ (where $t_j$ occurs in the displayed expressions of these "$\subseteq$" {Footnote 10}). The separation schema is $y_j \rightarrow x_j \vdash \forall . \forall x_j \exists y_j \forall \tilde{z} (\tilde{z} \varepsilon y_j \sim \tilde{z} \varepsilon x_j \ \&\ \phi(\tilde{z}))$ where $\phi(\tilde{z})$ does not contain $x_j$, $y_j$ freely and $\forall \tilde{z}$ stands for $\mathop{\&}\limits_{h=-1}^{k} \forall z_h$.

8. Let De be any formal proof or deduction in $\widetilde{ZF}_k$ (in particular, it may be $Ctd_\ell$ discussed earlier). For any $j = -1, 0, \ldots, k$, let $\underline{m}_j$ be a $N_j$-number such that $f(t_j) \leq m_j$ for any constant termoid which occurs in De. Let there be an assignment Asm of $N_j$-numbers $w\bar{x}_j$ to all variables $\bar{x}_j$ which occur in De. I shall call $w\bar{x}_j$ the "weight" of $\bar{x}_j$ (under Asm).

Let the "range" rg $\bar{x}_j$ be defined, for each variable $\bar{x}_j$ in De,

as the set $\{\underline{x}_j \mid f(\underline{x}_j) \leq \underline{m}_j + w\bar{x}_j\}$ of all events of $D^j$ which fit the condition $f(\underline{x}_j) \leq \underline{m}_j + w\bar{x}_j$. Only those events of $D^j$ shall be considered as "non-alien" to De (under the present consideration) which belong to rg $\bar{x}_j$ for some $\bar{x}_j$ in De. (Constant termoids $t_j$ in De must belong to the range of any $\bar{x}_j$ in De - and actually some variable $\bar{x}_j$ shall occur in De if some $t_j$ does. Notice also that by these definitions, the constant composition arrows contribute to the variable ranges and thus to the domains of the models.)

<u>Asm</u> will be called a "<u>normal</u>" assignment if, under <u>Asm</u>, any arrow $\vec{y}_j \rightarrow \phi(\bar{x}_j)$ (where $\phi$, if present, is a function symbol) which precedes an axiom in De is "<u>correct</u>", i.e. each value of its right side has to be a value of its left side).

For the other non-logical axioms (listed below), the preceding arrows can be shown to be correct. The union set axiom:

$$y_j \rightarrow hf(x_j) \vdash \forall x_j \exists y_j \forall \tilde{z}(\exists t_j(\tilde{z} \in t_j \;\&\; t_j \in x_j) \supset \tilde{z} \in y_j),$$

where $\forall \tilde{z}$ stands for $\mathop{\&}\limits_{h=-1}^{k} \forall z_h$ (and the arrow may be "weakened" at least to $y_j \rightarrow x_j$ ), $j = 0, \ldots, k$.

The Replacement axioms:

$$u_j \xrightarrow{2} z_j,\; y_j \rightarrow hf(z_j) \;\vdash \forall .\, \forall x_j \exists y_j \forall z_j(\forall z_j \forall u_j \forall \tilde{t}(\phi(u_j, \tilde{t}) \;\&$$

$$\phi(z_j, \tilde{t}) \supset u_j = z_j) \;\&\; \exists \tilde{t}(\tilde{t} \in x_j \;\&\; \phi(z_j, \tilde{t})) \supset z_j \in y_j)$$

where $\forall \tilde{t}$ stands for $\mathop{\&}\limits_{h=-1}^{k} \forall t_h$ and $\exists \tilde{t}$ for $\bigvee\limits_{h=-1}^{k} \exists t_h$ and $\phi(z_j, \tilde{t})$ does not contain $x_j$, $y_j$ freely.

The axioms:

$$\forall \tilde{z} \forall \alpha \neg \tilde{z} \in \alpha ,$$

where $\forall \tilde{z}$ stands for $\mathop{\&}\limits_{h=-1}^{k} \forall z_h$ and $\alpha$ is a variable of kind $-1$;

(#) $\quad \forall z_j \forall x_i \neg z_j \in x_i \quad$ for $-1 \leq i < j \leq k$;

(#) $\quad \forall x_j \forall \tilde{z}(\tilde{z} \in x_j \supset \tilde{z} \in hf(x_j))$

where $\forall \tilde{z}$ stands for $\mathop{\&}\limits_{h=-1}^{k} \forall z_h$ , $j = 0, \ldots, k$.

The infinity axioms:

$$y_0 \to I^0 \vdash \exists y_0 \ \forall \alpha (\alpha \ \varepsilon \ y_0)$$

and

$$y_j \to I^j \vdash \exists y_j \mathop{\&}_{h=-1}^{j-1} \forall x_h (x_h \ \varepsilon \ y_j),\ j = 0, \ldots, k,$$

where $I^j$ stands for $\bigcup(a_0^j, \ldots, a_{z_j}^j)$ (whence $f(I^j) = 0 \leq m_j$ and the arrows are "correct" under any choice of Asm).

$\widetilde{ZF}_k$ is now described. The separation axioms entitle one to replace the "main" $\supset$ in the rest of the "comprehension axioms" by $\sim$ ; the axioms of pair are not listed because of their redundancy given the replacement schema, although the union set and infinity axioms are also used to "get" the "oversets" for the pair $\{x_i, y_j\}$, $i < j$, or $\{\alpha, \beta\}$. The union set axioms together with $\forall z_j \forall x_i \neg z_j \ \varepsilon \ x_i$, $\forall \tilde{z} \forall \alpha \neg \tilde{z} \ \varepsilon \ \alpha$ entitle one to "tilda-ize" all Latin variables in these axioms except for (#) (in the power set and the replacement axioms with $\exists y_j$, one does this only to such $x_h$ or $z_h$, respectively, in which $h \leq j$ . That is the form to be used for the model for $\widetilde{ZF}_j$ and the consideration of its $On_j$ as the "inaccesible" aleph, $j = 0, \ldots, k-1$). The restriction on $x_j$ in the schemata of Separation and Replacement actually does not narrow the formal provability.

Only normal assignments Asm will be considered in what follows. There is the possibility of satisfying the union and power set axioms by $y_j$ of "type" higher than $x_j$ if, for $j = 0, \ldots, k$, the "comprehension" axioms are checked in the usual way in accordance with type theory; the condition $y_j \to hf(x_j)$ expresses that (see the formulas (*) and (0)). For the Separation axioms, the condition $y_j \to x_j$ entails that any subset $y_j$ of $x_j$ - in particular, any value of $y_j$ in the axiom - has the degree $\leq$ that of $y_j$ (because its degree is $\leq$ that of $x_j$), and thus the axioms hold. Here I drop this checking for the Replacement axioms (which are beyond the type theory); but for the "axiom of pair" the checking is still available, the arrows being $y_j \to hf(x_j)$, $y_j \to hf(u_j)$ for the "pair" $\{x_j, u_j\}$ and $y_j \to hf(x_j)$, $y_j \to I^j$ for the pairs $\{x_j, u_i\}$ or $\{u_i, x_j\}$, $i < j$. In the case of $x_{-1}$, $u_{-1}$, the pair shall be some $y_0$ and the arrows - $y_0 \xrightarrow{2} I^0$).

The demonstroids found for the axioms of $\widetilde{ZF}_k^i$ (see [4] and p.23 and the Appendix of pp 36–37 below) also apply to the stronger system $\widetilde{ZF}_k$ and yield just the arrows above for the axioms. That suffices also for the purpose of this checking.

9. The logical axioms of the WPC are satisfied under any assignment Asm because they are expected to hold for arbtrarly chosen finite ranges of their variables {Footnote 11}. In order, therefore, to satisfy all logical axioms of De, it suffices to satisfy the strong Bernays axioms i.e. to make their arrows $\bar{y}_j \rightarrow \phi(\bar{x}_j)$ correct. It is important to notice that if a variable occurs in the right side of an arrow (in which case the arrow is called "variable"), then it has the same kind j as the left side. That is simply a property of the formal system under consideration {Footnote 12}.

Besides that, in order that an arrow $\bar{y}_j \rightarrow \phi(\bar{x}_j)$ be correct, it suffices (by (0) and (*) above) that the following "weight-arrow condition" (w.a.c.) holds:

$$\text{(a)} \quad w\bar{x}_j < w\bar{y}_j$$

If $\phi$ is void i.e. the arrow has the shape $\bar{y}_j \rightarrow \bar{x}_j$, then this condition can be weakened to

$$\text{(b)} \quad w\bar{x}_j \leq w\bar{y}_j$$

Now in order to get a finite model for De, it suffices to find such an assignment Asm that the w.a.c.(a) (or, respectively,(b)) is satisfied by all arrows in De. (Such an assignment must be, in particular, normal).

10. However, that is possible only if De satisfies a restriction, or belongs to one of two "fragments" of $ZF_k$ I am going to describe.

Firstly, the w.a.c. (a) cannot be satisfied by an assignment of weights $w\bar{x}_j$ if the arrows of De contain a "C-cycle": $\bar{x}^1 \rightarrow \phi^1(\bar{x}^2)$, $\bar{x}^2 \rightarrow \phi^2(\bar{x}^3), \ldots, \bar{x}^m \rightarrow \phi^m(\bar{x}^1)$. If all $\phi^1, \ldots, \phi^m$ are void, then the C-cycle is not "dangerous" because (b) can be used instead of (a) - and in order to satisfy the arrows of the C-cycle it suffices to put $w\bar{x}^i = 0$ for $i = 1, \ldots, m$. But if at least one $\phi^c$, $1 \leq c \leq m$, really occurs in the C-cycle, the latter will be an irretrievable obstacle for finding an assignment Asm which satisfies the w.a.c. In this case the C-cycle will be called "dangerous". Unless De (i.e. the arrows of De) contains a dangerous C-cycle, there is an easy way to decompose them in the "maximal C-strings"

$$\bar{x}^1 \rightarrow \phi^1(\bar{x}^2),\ \bar{x}^2 \rightarrow \phi^2(\bar{x}^3),\ \ldots,\ \bar{x}^m \rightarrow t$$

where $t$ does not contain $\bar{x}^{1}$ and, possibly, some non-dangerous C-cycles. For simplicity's sake, let these non-dangerous cycles be dropped (they create only minor problems). Then the weights can be assigned, originally, to the variables of each C-string by following them in the right-to-left direction and assigning, at each step, the smallest weight compatible with the w.a.c. Then, if there is more than one maximal C-string in this decomposition, only a dangerous C-cycle can prevent the possibility of modifying these assignments in the different strings so that each variable shall have the same weight assigned.

11. Let us consider the fragment of $\widetilde{ZF}_k$ consisting of the formal proofs without dangerous C-cycle; call it $\widetilde{ZF}_k^{wdc}$ . The rules of inference are

$$\text{(MP)} \qquad \frac{\Gamma \vdash A \qquad\qquad \Delta \vdash A \supset B}{\Gamma \quad \Delta \vdash B}$$

and the conjunction rule

$$\text{(cr)} \qquad \frac{\Gamma_1 \vdash A_1 \quad \cdots\cdot \quad \Gamma_m \vdash A_m}{\Gamma_1, \ldots, \Gamma_m \vdash \& (A_1, \ldots, A_m)}$$

which will be accepted independently (although in other cases this is a redundant rule). If all $\Gamma_c$, $\Gamma_d$, $c, d = 1, \ldots, m$, are the same arrows $\Gamma$ it is allowed to put this $\Gamma$ below the bar only once (in which case the use of the rule will be referred to as $\hat{cr}$ ). The rule (cr) is useful in the consideration of the "mixed" quantifiers $\forall\tilde{x}$ used in some of the axioms of $\widetilde{ZF}_k$.

The last rule to be mentioned permits one to drop the arrows when all of them are "correct" but <u>only on the last step of a formal proof or deduction De</u> {Footnote 13}.

So, until the very last step each sentence "accepted" in De shall be preceded by arrows (if any at all occur; the WPC axioms have none). It is a solvable property of a sentence $\Gamma \vdash F$ whether the text $\Gamma$ of arrows "preceding" a formula $F$ contains a dangerous C-cycle. If $\Gamma$ contains no dangerous C-cycles then this property also holds for all sentences on which $\Gamma \vdash F$ is dependent in De.

This property of the deductions of the fragment $\widetilde{ZF}_k^{wdc}$ can be involved in the formulation of the predicate $B(m, n)$ {Footnote 14} of the Gödel theory with the preservation of the primitive recursiveness of that predicate. Let the formula be denoted by $B^{wdc}(m, n)$.

This restriction, i.e. not containing dangerous C-cycles, intrudes

if one is trying to prove in $\widetilde{ZF}_{k+1}$ the consistency formula $Con_{\widetilde{ZF}_k}$ for $\widetilde{ZF}_k$ (due to Tarski). I am unaware of another example in $\widetilde{ZF}_k$. Also, some paradoxes in the extended language (see [4, part VI]) are prevented just by dangerous C-cycles {Footnote 15}.

One of these paradoxes consists in the possibility of establishing not only the infinity but also the finiteness of $I^j$ ($j = 0, \ldots, k$). If the "Tarski argument" were formalizable without any dangerous C-cycle (and so far no theorem convincingly prevents that), then a simple finite model for a version of this paradox seems almost unavoidable.

Surely each axiom of $\widetilde{ZF}_k$ (including the strong Bernays axioms) is preceded only by arrows not containing dangerous C-cycles. The use of (cr) preserves the property (no variable in $\Gamma_c$ shall be identified with a variable in $\Gamma_d$, $d \neq c$.)

At the steps of (MP) this property may be lost. The danger is significant because each case of renaming of a bound variable

$$\bar{x} \rightarrow \bar{y} \quad \vdash \forall . \; \forall \bar{x} A(\bar{x}) \supset \forall \bar{y} A(\bar{y})$$

is the consequence of the strong Bernays axiom

$$\bar{x} \rightarrow \bar{y} \vdash \forall . \; \forall \bar{y} ( \forall \bar{x} A(\bar{x}) \supset A(\bar{y}))$$

(and similarly with $\exists$ ). On the other hand, the need for renamings can be reduced by suitable preparations in the text of the axioms of De. (The choice of variables in the axioms is restricted only by the general stipulations concerning the freedom of substitutions and the small restrictions above.) Also the theorems to be formally proved are specified only up to the "congruence" of the formulas. In addition to that, when a reasoning to be formalized admits a type theoretical formalization, then the "dangerous C-cycle" often are prevented. At least if the variables $\bar{x}^1, \ldots, \bar{x}^m$ of the cycle are of the kind $j \geq 0$, then the ranges correspond to the degrees (which correspond to the "types"), so that if the cycle is dangerous then $\bar{x}^m$ must possess the values of $\bar{x}^1$ which would force $\bar{x}^1$ to have a larger degree or type than it actually does. But this argument does not apply to the case of $j = -1$. However, the derivation of the Dedekindean infinity axiom can be performed without any dangerous C-cycle.

Any finite list of symbols for primitive recursive functions can be admitted in an extension of $\widetilde{ZF}_k$ and in De. The functions can be majorized by a strictly monotone unary function $\phi(\alpha) > \alpha$, and the ranges $rg\alpha$ for the variables $\alpha$ of the kind $-1$ have then to be rede-

fined as the sets $\{\underline{\alpha} | \underline{\alpha} \leq \phi^{w\bar{\alpha}} (m_{-1} + w\bar{\alpha})\}$ where the superscript $w\bar{\alpha}$ indicates the $w\bar{\alpha}$ -fold interation of $\phi$. The w.a.c. (a), (b) remain unchanged.

Thus, primitive recursive function theory can be imbedded in $\widetilde{ZF}_k^{wdc}$. This theory is "sufficiently non'circular" that it can be developed without dangerous C-cycle. In particular, both Gödel incompletness theorems can be proved for $\widetilde{ZF}_k^{wdc}$ (not withstanding the possibility of a finite model).

12. Probably more attractive than $\widetilde{ZF}_k^{wdc}$ is a narrower fragment of $\widetilde{ZF}_k$. For any part A of a formula F, let its "<u>depth</u>", dA, be defined as the number of free variables in A. If F is closed, then dA is the number of those quantifiers in F which bind in A (or are "<u>non-fictitious with respect to A</u>"). If A is $Q\bar{u}B$ where Q is $\forall$ or $\exists$, then the "<u>depth</u>" of this $Q\bar{u}$, or of this $\bar{u}$, is defined as dA. Now the idea is to define $w\bar{u}$ as the depth of the quantifier occurrence of $\bar{u}$. An obstacle to this definition is that the depth, $d\bar{u}$, may be different for different $Q\bar{u}$'s in De with the same $\bar{u}$. If this does not happen, a formula will be said to have the "equal depth property" (EDP).

In particular, it is easy to check that the w.a.c. is fulfilled for any arrow of non-logical axioms of $\widetilde{ZF}_k$ as well as for any logical axioms with an arrow $y_j \to x_j$ and with a constant arrow from the language of S'. (Such a constant arrow, unless covered, "survives" up to the "main" member of the r.s.t.) If the arrow of a strong Bernays axiom contains a function sybol, $\phi$ , the axiom must be reformulated as $\bar{y}_j \to \phi(\bar{x}_j) \vdash \forall . \forall \bar{x}_j (\forall y_j (\bar{x}_j = \bar{x}_j \ \& \ A(\bar{y}_j)) \supset A(\phi(\bar{x}_j)))$ (and similarly with $\exists \bar{y}_j$). Then $w\bar{x}_j < w\bar{y}_j$ follows. But now the strong Bernays axioms also have to satisfy EDP for which purpose some additional insertions of $\bar{y}_j = \bar{y}_j$ in $A(\bar{x}_j)$ can be needed if $A(\bar{y}_j)$ contains $\bar{x}_j$, or if a constant termoid is substituted for $\bar{y}_j$. The EDP is broken by the two $\forall z_j$ in the Replacement axioms as well as by $\forall \tilde{t}$, $\exists \tilde{t}$ in the axioms; again, that creates some problems which are, however, secondary (the identifications of these variables are of need only in order to get some arrows swallowed; but these arrows might be tolerated without any danger of a dangerous C-cycle).

The variables which are to be identified in the text of a logical axiom of $\widetilde{ZF}_k$ (or of a contradiction formula) in order to recognize the formula as such always have the same depth - with the exception of the "<u>explicit</u>" $\forall \bar{x}$ in the quantification axioms. For example, in the axioms

(Dis) $$\forall . \forall \bar{x}(B \supset C) \supset (\forall \bar{x}B \supset \forall \bar{x}C)$$

the $\bar{x}$'s may have different depths. This case can prevented by imposing the restriction that $\forall\bar{x}B$, $\forall\bar{x}C$ contain the same free variables. I call that the "both sides restriction" (BSR) {Footnote 16}; the right (left) side restriction, (RSR, (LSR), is that each variable free in $\forall\bar{x}C$ be free in $\forall\bar{x}B$ (or vice versa). The quantifiers in the closure string $\forall.$ may be fictitious and have no fixed order. Because of that, further axioms of quantification of the WPC can be reduced to the schemata $\forall.\forall\bar{x}A \supset A$, $\forall.A \supset \forall\bar{x}A$ where A does not contain $\bar{x}$ freely, the Weak Bernays Axioms $\forall.\forall\bar{x}B \supset B$ and the respective axioms for $\exists$ as well as $\forall.\forall\bar{x}(B \supset A) \supset (\exists\bar{x}B \supset A)$ where A does not contain $\bar{x}$ freely: but in the classical logic the $\exists$-quantifier is unnecessary. (notice that the axiom schema $\forall.\forall\bar{x}(A \supset B) \supset (A \supset \forall\bar{x}B)$ "dual" to the last one is redundant as soon as $\forall.A \supset \forall\bar{x}A$ and the schema (Dis) are postulated.) Now, these schemata too must be restricted to cases when $\bar{x}$, if it appears in the schema in two distinct non-fictitious quantifiers, be of the same depth; for the Weak Bernays Axioms, $\forall.$ must not terminate by $\forall\bar{x}$.

It suffices to impose the "equal depth restriction" (i.e. the fulfillment of the E.D.P.) - or the B.S.R. - only on the variables $\bar{x}$ which occur in the variable arrows in De (because for the rest of the variables, $\bar{u}$, the axioms shall be fulfilled if $w\bar{u}$ is defined as 0).

The equality axioms of the WPC are $\forall.\ r = s \supset (A(r) \supset A(s))$ where A(r) is atomic and - for the fragment at issue - has the same depth as A(s); r, s are any termoids of the same kind. Further equality axioms $\forall\bar{x}(\bar{x} = \bar{x})$, $\forall\bar{x}_i\forall\bar{y}_j[\neg\bar{x}_i = \bar{y}_j], (i \neq j)$ as well as the propositional axioms $\forall.P$ (where P is any instance of a usual schema) are subject to no restrictions.

The strong Bernays axioms

$$\bar{x} \to t \vdash \forall.\ \forall\bar{x}A(\bar{x}) \supset A(t),\quad \bar{x} \to t \vdash \forall.A(t) \supset \exists\bar{x}A(\bar{x})$$

with a constant termoid t shall be subject to the restriction that the free $\bar{x}$ in $A(\bar{x})$ does not occur in $A(\bar{x})$ in the scope of a non-fictitious {Footnote 17} quantifier with avariable $\bar{u}$ which occurs in a variable arrow. This fragment shall be denoted by $\widetilde{ZF}_k^{BSR}$. It is easy to see that its formal deductions De contains no dangerous C-cycle.

13. The logic of the system $\widetilde{ZF}_k^{BSR}$ probably does not allow the proof of some formulas with EDP provable in the WPC as for example: $\forall.\forall\bar{x}(A \& B) \sim (\forall\bar{x}A) \& B$ where B contains freely the same variables as $\forall\bar{x}A$. Let such formulas be postulated as soon as the need arises. They are, in any case, enumerable. The fragment $\widetilde{ZF}_k^{BSR}$ thus

extended still is contained in $\widetilde{ZF}_k^{wdc}$. Let it be denoted by $\widetilde{ZF}_k^{ed}$ ("ed" from "equal depth").

That is, however, not a full logic. In particular the formulas $\forall \bar{x} \forall \bar{y} A \supset \forall \bar{y} \forall \bar{x} A$ are not, in general, available in $\widetilde{ZF}_k^{ed}$. But the sentences

$$\bar{x} \to \bar{u},\ \bar{y} \to \bar{v}\ \vdash \forall \bar{x} \forall \bar{y} A(\bar{x}, \bar{y}) \supset \forall \bar{v} \forall \bar{u} A(\bar{u}, \bar{v})$$

can still be available without dangerous C-cycles, although this fact must be checked for each use. In general, the formulas $\forall . \forall \bar{x}(B(\bar{x}) \supset C(\bar{x})) \supset (\forall \bar{u} B(\bar{u}) \supset \forall \bar{v}\ C(\bar{v}))$ can be used instead of $\forall . \forall \bar{x}(B(\bar{x}) \supset C(\bar{x})) \supset (\forall \bar{x} B \supset \forall \bar{x} C)$ when the B.S.R. is broken. They yield the latter axiom with the help of the arrows $\bar{x} \to \bar{u}$, $\bar{v} \to \bar{x}$ (in these arrows, $\bar{x}$ refers to its second and third quantifier appearances in the axiom). If the axiom fits the R.S.R., then $\bar{u}$ can be chosen as $\bar{x}$ and the arrow $\bar{x} \to \bar{x}$ fits (b) of the w.a.c.; hence the axiom is still available as a theorem of $\widetilde{ZF}_k^{ed}$.

The provisos for the strong Bernays axioms without variables in t can be weakened. In any case, these provisos don't hinder the proof of the Gödel incompletencess theorems for $\widetilde{ZF}_k^{ed}$.

In many cases the restrictions of the BSR can be overcome by appending conjunctively to B the equalities $\bar{u} = \bar{u}$ with the variables $\bar{u}$ missing in B. However, this device does not help to overcome some of the difficulties which arise for the procedure of elimination of definite descriptions (there are no such difficulties for $\widetilde{ZF}_k^{wdc}$).

14. So far, I have considered the logic for a particular formal proof De. (Instead of being such a proof, De might be also a formal deduction.) Now, I am going to extend this consideration. Axioms of $\widetilde{ZF}_k$ admit proofs formalizable with the help of the rule $(\underset{\sim}{Ca})$ and representable as r.s.t.'s. Most of the axioms in De start with non-fictitious $\forall$-quantifiers. The only exceptions are i) closed instances of the logical axioms with the void $\forall .$; ii) axioms starting with the fictitious $\forall$; iii) the "infinity axioms" (page 16 ) starting with $\exists y_j$, $j = 0, \ldots, k$ - and preceded by the arrows $y_j \to I^j$; they are easily deducible, however, from the formulas $\forall x_h (x_h \in I^j)$ $(h = -1, 0, \ldots, j - 1)$ which start from the non-fictitious $\forall x_h$. In the cases iii) and, partially, ii), the axioms of De are easily deducible from formulas starting with non-fictitious $\forall$-quantifiers - I shall include these formal deductions in De, and the resulting formal proof (or deduction from hypothesis HS which I shall not mention any

more) shall be denoted by De*. The hypothesis (besides those of HS) shall be the $\underset{\sim}{Ca}$ conclusions; now, I shall join to De* the presentations of these $\underset{\sim}{Ca}$'s (to be placed just above the $\underset{\sim}{Ca}$ conclusions) and thus obtain the "main" member of the r.s.t. to be called the De-r.s.t. The members 1-subordinated to it shall be those for these $\underset{\sim}{Ca}$ conclusions; they shall be assigned the '<u>height</u>' 1 and can contain further $\underset{\sim}{Ca}$ conclusions whose premises shall be "yielded" by the r.s.t. members of the '<u>height</u>' 2, and so on.

The language of the De-r.s.t. shall contain, besides = and $\varepsilon$ (the binary predicates applicable to termoids of any kind) also the binary predicate $h_j < n_j$ applicable only to events $a_0^j$ , $[a_0^j]$ , $[[a_0^j]]$, ... of $D^j$ which form a Nn, $N^j$ isomorphic to $N_j$, as well as to the termoids $f(r_j)$ having them as their values, with any constant termoid $r_j$ ($j = 0, 1, \ldots, k$).

The system $\widetilde{ZF}_k$ shall be replaced by the equiconsistent system $\widetilde{ZF}_k^i$ whose logic is the intuitionistic version of the logic of $\widetilde{ZF}_k$; the logical operators $\exists, \vee$ are replaced, in the non-logical axioms of $\widetilde{ZF}_k$, by their usual "translations" $\neg\forall\neg$ , $\neg\&\neg$ (denoted by $\underset{=}{\exists}$ , $\underset{-}{\vee}$, respectively), and some instances of $\forall . \neg\neg P \supset P$ with atomic $P$ are postulated in $\widetilde{ZF}_k^i$ as the non-logical axioms; actually, it suffices to use in that way only the instances with $x_j \varepsilon y_j$ ($j = o, \ldots, k$) as the P. The consistency of $\widetilde{ZF}_k$ relative to $\widetilde{ZF}_k^i$ is provable by means of the translations above (cf. [6, §81]) and the translations of $r_i \varepsilon s_j$, $i < j$, as $\neg\neg r_i \varepsilon s_j$.

De shall be considered as a formal proof (or deduction) in $\widetilde{ZF}_k^i$, and the De-r.s.t. shall be a r.s.t. with the intuitionistic logic. The covering rules shall be specified as

$$\text{(Cv)} \qquad \frac{f(r_j) \leq f(s_j) \qquad \bar{x}_j \to r_j, \Gamma \vdash F}{\bar{x}_j \to s_j, \Gamma \vdash F}$$

and

$$(\text{Cov}_j) \qquad \frac{\bar{x}_i \to r_j \cup v_j, \Gamma \vdash F}{\bar{x}_j \to r_j, \bar{x}_j \to v_i, \Gamma \vdash F}$$

The only axioms, besides the logical ones, with the void $\forall$.'s (if any), shall be the closed atomic formulas, or their negations, which are "true" in virtue of the calculations based on the definitions of the functions and predicate symbols of this system.

The formal system thus described shall be denoted by S'. Also a kind, S*, of the metathory of S' shall be at issue. The parameters

$\underline{x}_j, \underline{y}_j, \ldots$ for the values of variables shall belong to the language of S*, as well as the notations $\mathfrak{z}_j, \mathfrak{z}_j', \mathfrak{z}_j'', \mathfrak{z}_j^c$ $(c = 1, \ldots, m)$, $a_h^j$ for the kernels (for $j = -1$, the bold case Greek letters $\underline{\alpha}$, $\underline{\beta}$, ... shall be used instead of $\underline{x}_{-1}, \underline{y}_{-1}, \ldots$, h of $a_h^j$ shall be considered as a parameter of S*; other small case Latin letters, like g, f,..., can be used instead. Occasionally they may bring superbars and the like; also constant figures like 0, 1,..., and $\underline{z}_j$, as well as the termoids $\nu^j(\underline{x}_i)$ of $\nu^j(\underline{\alpha}_g)$ - see p. 11 - can occur at the place of h in $a_h^j$). This language - together with some common notations like the dots in $\bigcup(\mathfrak{z}_j, \ldots, \mathfrak{z}_j^m)$ or elsewhere - shall be used in the $\underset{\sim}{Ca}$ presentations. Only three sorts of the latter shall be allowed in S*, viz, :

a) the "<u>parametrical</u> $\underset{\sim}{Ca}$'s" - referred to as the Ca's - are presented just by the schema $P(\tilde{x})$ of the deductoids $P(\underline{x})$ of their $\underline{x}$-premises; $P(\underline{x})$ is to be obtained from the schema by the substitution of $\underline{x}$ for $\tilde{x}$;

b) the "<u>recursive</u> Ca's" - referred to as the rCa's - which can be examplified by the displayed obtaining of the demonstroid of the $\underline{x}_j$-premise in the case of the usual "mathematical induction", (cf[3, p.10]). The presentation contains two schemata corresponding to the base and to the "inductive step". When $j = 0, \ldots, k$, then the "induction" is actually going on on the number m in the presentation of $\underline{x}_j$ as $\bigcup(\mathfrak{z}_j^1, \ldots, \mathfrak{z}_j^m)$ without repetition of the kernels: the mentioned two schemata can fall into subschemata corresponding to the case when $\mathfrak{z}_j^1$, or $\mathfrak{z}_j^{h+1}$ has the shape $[\underline{u}_j]$ or $a_h^j$ (with a further subdivision corresponding to the cases $h = 0$, $h \neq 0$, respectively);

c) the "<u>simple</u> Ca's" - referred to as SCa's - have, in the "simplest" cases, the $\underline{x}_j$-premises $B(\underline{x}_j)$ without any occurrence of variables. They can be obtained by the propositional demonstroids from the axioms $A(\underline{x}_i, \underline{y}_h)$ which are atomic formulas or their negations. These demonstroids can be presented by a small number of schemata corresponding to a few cases, solvable in terms of the definitions of the atomic predicates and function symbols.

In a particular case - on which the infinity axioms $y_j \rightarrow I^j \vdash \exists y_j \forall^{j-1}\tilde{x}(\tilde{x} \in y_j)$ depend - the rCa conclusions are $\forall x_i(x_i \in I^j)$ and the $\underline{x}_i$-premises $\underline{x}_i \in I^j$, $i < j$, are just the axioms of S' ($\forall^{j-1}\tilde{x}$ stands for $\overset{j-1}{\underset{h=-1}{\&}} \forall x_h$).

The conclusions of these rCa's may still contain "parameters" $\underline{u}$, $\underline{v}, \ldots$ of different kinds which can be bound by the quantifiers of further $\underset{\sim}{Ca}$'s. These $\underset{\sim}{Ca}$'s - applied consecutively one by one till these parameters are exhausted (without interference of other logical steps - except those by cr in order to introduce the quantifiers $\forall \tilde{z}$) - are

also referred to as the SCa's.

The ranges shall be assigned to occurrences of the variables in the main r.s.t. member just as it was described for De above. The choice of the number $m_j$ shall be always referred to the "main" member. The variables in the h-subordinated members (h > 0) can mostly be linked, for any occurrence, by a string of identifications used at the logical steps, with a variables in the main member. They will be assigned the same "weights" as the latter. In instances of the quantification axioms, the identifications of two different $\bar{x}$'s at such an axiom shall be exempted from these strings. In cases when there is no such string, the r.s.t. can easily be transformed by means of the axioms of the WPC- not necessarily fitting the EDP - into one in which the strings always exist. Here it suffices to introduce, when some A of MP disappears, the conjunction members $\forall\bar{u}(\bar{u} = \bar{u})$ with the disappeared $\bar{u}$ into the conclusion B of the MP, and to perform, when needed, similar transformations in the $\underset{\sim}{Ca}$ conclusion. In addition, the "induction quantifier" $\forall\bar{x}$ of a $\underset{\sim}{Ca}$ conclusion can always be made to have the weight $\leq$ than $\bar{x}$ at each swallowed arrow (because the latter are to be worried about only if their $\bar{x}$ occurs in $A(\bar{x}_j)$ in which case the parts $Q\bar{x}B$ of this formula can be replaced by $\forall\bar{u}^1 \dots \forall\bar{u}^c\, Q\bar{x}(B\ \&\ \bar{u}^1 = \bar{u}^1 \dots \&\ \bar{u}^c = \bar{u}^c)$. The $\underset{\sim}{Ca}$ conclusion may be, e.g. a (SBA), or equality axiom, with $\forall$..

The weights $w0c_{\bar{x}}$ can be assigned, to quantifier occurrences $0c_{\bar{x}}$ of variables - and to those bound by them - also as follows: let $0c_{\bar{x}}$ occur at a r.s.t. member Q h-subordinated to the "main" one, and let it have the depth d (i.e. let d be the number of free variables in the formula starting with that quantifier). Then the weight, $w0c_{\bar{x}}$, be defined as h + d. The occurrences of variables in arrows get their weights accordingly. (That is an assignment of the nature mentioned on p. 8 ; it seems in some respects more convenient than the work with the chains of identifications, and still sufficient for the considerations of the r.s.t.'s at issue.)

When a $\underset{\sim}{Ca}$ conclusion formula, $\forall\bar{x}A(\bar{x})$, contains a part $Q\bar{u}B(\bar{u})$ not containing $\bar{x}$ freely where $Q\bar{u}$ is a non-fictitious quantifier, then each such part has to be replaced, in the $\underset{\sim}{Ca}$-conclusion, by $Q\bar{u}(B(\bar{u})\ \&\ \bar{x} = \bar{x})$; the former $\underset{\sim}{Ca}$-conclusion can then be restored by means of the WPC (not necessarily within its part B.S.R. or EDP!). The respective transformations for the $\underset{\sim}{Ca}$-premises are always available in the WPC with the B.S.R. This transformation has to be applied to all $\underset{\sim}{Ca}$'s used in the demonstroids of the non-logical axioms of $\widetilde{ZF}^i_k$.

In the r.s.t. transformed in that way the corresponding occurrences of variables in the premise and conclusion formulas of the $\underset{\sim}{Ca}$'s have

the same weight. This argument does not apply, however, to the Ca's yielding the logical axiom Dis; in these Ca's the variable $\bar{x}$ in each of the three $\forall\bar{x}$'s has to be assigned in the premise the same weight as in the conclusion (and, eventually, as in the occurrence of the axiom Dis in De).

The Ca's of these three kinds shall be called "normal". Ca's shall be considered as used in a r.s.t. member, Q, which contains its presentation (as a top of the member tree figure), and also in other members of the r.s.t. to which this Q is h-subordinated, $h > 0$. A r.s.t. shall be called normal if only normal Ca's will be used in its members.

When a r.s.t. is normal, its presentation is actually determined (a) by its "main" member, P (which is just displayed or otherwise presented); and (b) the assignment of the ranges to all occurrences of its variables (as the "induction variables" of the Ca's). Namely - each Ca presentation in the "main" member together with any value $\underset{\sim}{x}$ of the "induction variable" $\bar{x}$ of the Ca determines the 1-subordinated (to P) member $P^1(\underset{\sim}{x})$; in the same way, the Ca presentations in $P^1(\underset{\sim}{x})$ - together with the values $\underset{\sim}{u}$ of their "induction variable" $\bar{u}$ - shall determine the members $P^2(\underset{\sim}{u}, \underset{\sim}{x})$ 2-subordinated to P, etc. The maximal "rank" h of a member $P^h(\underset{\sim}{w}, \ldots, \underset{\sim}{u}, \underset{\sim}{x})$ h-subordinated to P, $h \geq 0$, shall be called the height of the r.s.t.

For any normal r.s.t., its height shall be determined by its presentation (and it actually is independent of any specific choice of the assignment of the ranges.) The presentation shall also determine the maximal height of the construction tree (from atomic formulas) of a formula at a member of the r.s.t., as well as the "depth" of each quantifier at the "main" member and the "weight", $w0c_{\bar{x}_j}$, assigned to any clearly indicated occurrence of a variable $\bar{x}_j$ at a r.s.t. member.

$\underset{\sim}{w}_j$ shall denote the maximal $w0c_{\bar{x}_j}$ of an occurrence $0c_{\bar{x}_j}$ of a variable $\bar{x}_j$ at any member of the r.s.t. For any normal r.s.t., $\underset{\sim}{w}_j$ shall be determined by its presentation.

REMARK. The use of the Ca's sometimes helps in dispensing with some arrows which otherwise can cause dangerous C-cycles. Say, the passage from $\forall\bar{x}\,\forall\bar{y}B(\bar{x}, \bar{y})$ to $\forall\bar{y}\,\forall\bar{x}B(\bar{x}, \bar{y})$ can be performed as follows:

$$\begin{array}{rlll}
 & 0. & \forall \bar{x} \forall \bar{y} B(\bar{x}, \bar{y}) & \text{- hypothesis} \\
\bar{x} \to \underset{\sim}{x} \vdash & 1. & \forall \bar{y} B(\underset{\sim}{x}, \bar{y}) & \text{- 0; strong Bernays axiom, MP} \\
\bar{x} \to \underset{\sim}{x},\ \bar{y} \to \underset{\sim}{y} \vdash & 2. & B(\underset{\sim}{x}, \underset{\sim}{y}) & \text{- 1; strong Bernays axiom, MP} \\
\bar{y} \to \underset{\sim}{y} \vdash & a0. & \forall \bar{x} B(\bar{x}, \underset{\sim}{y}) & \text{- 2, Ca; the arrow } \bar{x} \to \underset{\sim}{x} \text{ swallowed} \\
 & b0. & \forall \bar{y} \forall \bar{x} B(\bar{x}, \bar{y}) & \text{- a0, Ca; the arrow } \bar{y} \to \underset{\sim}{y} \text{ swallowed.}
\end{array}$$

Now a version of the "deduction theorem" yields $\forall \bar{x} \forall \bar{y} B(\bar{x}, \bar{y}) \supset \forall \bar{y} \forall \bar{x} B(\bar{x}, \bar{y})$ - without arrows and without identifications of $\bar{x}$ and $\bar{y}$ in these quantifiers. (These identifications are not required for the purposes of following strings of identifications throughout different r.s.t. members.) However, the ranges of $\bar{x}$ (and of $\bar{y}$) are identifiable.

15. Now, it is natural to modify the language of the r.s.t. so that the weight superscripts shall be explicitly attached to the variables. So, the variables shall be $x_j^b$, $y_j^c$, ... - sometimes denoted by $\bar{x}_j^b$, $\bar{x}_j^c$, $\bar{y}_j^c$, ... where the bar indicates the abstraction from the choice of the concrete Latin letter.

The identifications of variables in formulas and arrows shall presuppose the identifications of these superscripts. The axioms of quantification are

$$\begin{array}{ll}
(\mathrm{Dis}^{bc}) & \forall\, .\, \forall \bar{x}^{\max(b,\ c)} (B \supset C) \supset .\ \forall \bar{x}^b B \supset \forall \bar{x}^c C, \\
(\mathrm{Fi1}) & \forall\, .\, \forall \bar{x}^c A \supset A \\
(\mathrm{Fi2}) & \forall\, .\, A \supset \bar{x}^c A \\
(\mathrm{WBA}) & \forall'\, \forall \bar{x}_j^c\, \forall''\, .\, \forall \bar{x}_j B \supset B
\end{array}$$

where A (in Fi1, Fi2) does not contain $\bar{x}$ freely. (I drop the superscripts within the formulas because they are determined by the superscripts in the quantifiers; so, if in $(\mathrm{Dis}^{bc})$ $b < c$, then $\forall \bar{x}^{\max(b,\ c)} (B \supset C)$ is $\forall \bar{x}^c (B(\bar{x}^c) \supset C(\bar{x}^c))$ whereas $\forall \bar{x}^b B$ is $\forall \bar{x}^b B(x^b)$; in the arrows, however, the superscripts shall be kept.) The axioms $(\mathrm{Dis}^{bc})$ don't cover the case when all three $\bar{x}$'s have different weights - but they imply (with the aid of (Res), p.29 ,and the chain inferences) these instances of the distributivity axioms and thus make them redundant.

These superscripts indicate the weights {Footnote 18} - which in the "main" r.s.t. member are the depths - and are $\leq w_j$ for each variable $\bar{x}_j$ of the kind j. In $(\mathrm{Dis}^{bc})$, different "explicit" occurrences of $\bar{x}$ can get different superscripts. In the closure strings $\forall$ or the strings $\forall'$, $\forall''$ of the closure string $\forall' \forall \bar{x}_j^c \forall''$ in the (WBA), the variables $\bar{x}_h$ may have any "admissible" weight (i.e. any superscript $\leq w_h$.)

Also in the instances of $\underset{\sim}{\mathrm{Ca}}$ the "induction variable" $\bar{x}_j$ can be assigned any superscript $\leq w_j$. The axioms $(\mathrm{Dis}^{bc})$, (Fi1), (Fi2), (WBA)

characterize the part WPC of the predicate calculus; in addition, the closures of the propositional and equality axioms belong to WPC. Here I describe only the postulates concerning the $\forall$- quantifiers. The strong Bernays axioms (SBA) shall have the shape

$$\bar{x}_j^c \rightarrow t_j \vdash \forall . \forall \bar{x}_j^c A(\bar{x}_j) \supset A(t_j)$$

with the usual restrictions on the substitutions in $A(\bar{x}_j)$. If the right side termoid $t_j$ in the arrow contains a variable, $\bar{y}_j$ - it must be assigned a superscript b such that the arrow is correct; then $\forall \bar{y}_j^b$ shall be the last quantifier in this $\forall$. . (I now drop the (SBA)'s with $\exists$ .)

The (SBA)'s can be formally deduced, in the WPC, from the "equality substitution axioms"

$$\text{(E-S)} \quad \bar{x}_j^c \rightarrow t_j \vdash \forall . \exists \bar{x}_j^c (\bar{x}_j^c = t_j)$$

(hint: use the equality axiom $\forall\forall \bar{x}_j^c . \bar{x}_j = t \supset [( \forall \bar{x}_j^c A(\bar{x}_j^c) \supset A(\bar{x}_j^c)) \supset ( \forall \bar{x}_j^c A(\bar{x}_j^c) \supset A(t))]$ and apply the interchanging of the antecedents, the (WBA) and the axiom for $\exists$ ). If $\exists$ is replaced by $\exists$ througout the logic, the same replacement shall be done in the (E-S)'s. In this case also $\neg\neg A(t) \supset A(t)$ is to be used - and it shall be available (cf [6, §81], Lemma 43A)).

In the modified systems S', S* - which I shall denote by $\underset{\sim}{S}'$, $\underset{\sim}{S}^*$, respectively - the axioms in question are the closed formulas with the void closure strings. The logical axioms with the non-void $\forall$'s shall be available by means of Ca's. So, the equality-substitution axioms of $\underset{\sim}{S}'$, $\underset{\sim}{S}^*$ take on the shape

$$\text{(E-S)} \quad \bar{x}_j^c \rightarrow t_j \vdash \exists \bar{x}_j^c (\bar{x}_j^c = t_j)$$

where $t_j$ is a constant termoid. Notice that such a statement can be considered as true even if the arrow is false. A Ca yields the form of the axiom above - to be used in the predicate calculus. If $\exists$ is removed from the logic, then the (E-S)'s are to be used with $\exists \bar{x}_j^c$, i.e. with $\neg\forall \bar{x}_j^c \neg$ .

The arrows must be correct in the (E-S)'s which occur in the r.s.t. members in order to keep the possibility to drop them, by $\overrightarrow{\text{MP}}$, at the last step of the "main" member. (But also the (E-S)'s and other $\underset{\sim}{\text{Ca}}$ conclusions with false arrows can be mentioned for other purposes -

such as justifications of occurrences of logical postulates or of some particular steps of identifications.)

If the part of the WPC with B.S.R. is at issue, then the axioms $(\mathrm{Dis}^{bc})$ are available only with $b = c$, whereas the rest of the axioms: (Fi1), (Fi2), (WBA) (for the $\forall$ -quantifiers) keep their schemata. So, the axioms of quantification shall be

$$\text{(Dis)} \quad \forall \bar{x}^c(B \supset C) \supset (\forall \bar{x}^c B \supset \forall \bar{x}^c C),$$

$$\text{(Fi1)} \quad \forall \bar{x}^c A \supset A,$$

$$\text{(Fi2)} \quad A \supset \forall \bar{x}^c A,$$

$$\text{(WBA)} \quad \forall' \forall \bar{x}^c \; \forall''.\, \forall \bar{x}^c B \supset B.$$

(Here I have written them without $\forall$.; but in the "main" member of a r.s.t. they can have it.) Only (Dis) is hereby weakened. In order to regain the strength of the WPC, only two further axioms suffice, viz.:

$$\text{(Res)} \quad \forall \bar{x}^c B \supset \forall \bar{x}^b B,$$

$$\text{(Ext)} \quad \forall \bar{x}^b B \supset \forall \bar{x}^c B$$

where, in both cases, $b < c$. (In the "main" member of the r.s.t. - and in the WPC, in general - the instances of these axioms may get the non-void $\forall$.) Vice versa - by taking in $(\mathrm{Dis}^{bc})$ B as C the axioms (Res) and (Ext) become available, with the aid of $\forall \bar{x}^b(B \supset B)$ or $\forall x^c(B \supset B)$ and MP, from the instances of $(\mathrm{Dis}^{bc})$ with $c < b$ or $b < c$, respectively.

The passage from $\forall \bar{x} \forall \bar{y} B$ to $\forall \bar{y} \forall \bar{x} B$ described in the remark at the end of the last section can be repeated in $\underline{S}^*$, with $\bar{x}, \bar{y}$ replaced by $\bar{x}^c, \bar{y}^d$. That is the content of the identifiability of two ranges mentioned at the end of the remark.

Now, the r.s.t. for the consistency proofs can be formulated in $\underline{S}^*$.

The axioms $(\mathrm{Dis}^{bc})$ with $b < c$, or, instead, the axioms (Ext), with one of their explicit variables $\bar{x}^b_j$, $\bar{x}^c_j$ occurring in a variable arrow, will actually occur only in the main r.s.t. member and, besides that, only as instances of the axioms by the same schemata, with the same b, c in the subordinated members used for these axioms in the main member. In the uppermost of these subordinated members they are without $\forall$. s - say, (Ext) has the shape $\forall \bar{x}^b B \supset \forall \bar{x}^c B$. These closed formulas can now be considered as the hypothesis of the r.s.t. {Footnote 19}.

Part III

1. For the purpose of the consistency proof for $ZF_k$, I used in [4] the model with (the) $k + 2$ infinite processes $D^j$, $j = -1, \ldots, k$. These processes were used as the ranges of the variables of different kinds. The problem of dangerous C-cycles did not arise - because they are "dangerous" just for the finite assignment which was there not at issue. So, the deductoid used for the "main" member of the r.s.t. might contain any C-cycles and there were no restriction on the quantification postulates. Other problems were important in view of the ultra-intuitionistic criticism - and the prototheories yielded solutions which in a sketchy form have been considered in [4]. Since then, they were given an essentially more constructive form and I intend to prepare that for publication.

In what follows, I shall consider the finiticized approach to the consistency proof. The deep problems raised by the ultra-intuitionistic criticism are not answered just by the finitization, even if the latter is completed. Actually - where is a proof that excludes the possibility that one day a paradox of the sort mentioned on p.19 will be presented, without any dangerous C-cycle or even with the WPC restricted by B.S.R.? That is the reason why I avoid referring to these models as to the consistency proofs - even for the fragments of logic I have considered above, and instead use the term "half-consistency proof" - or, even better, the consistency half- proof.

Now, in order to half-prove the consistency of $\widetilde{ZF}^i_k$, it remains to involve the axioms (Ext) in these half-proofs. For that end I have to change the interpretations of the non-fictitious quantifiers $\forall \bar{x}^c_j$, $\exists \bar{x}^c_j$ whose variables occur in the variable arrows of the r.s.t. I shall interpret these $\exists \bar{x}^c_j$ as $\neg \forall \bar{x}^c_j \neg$ and now dicuss only the case of $\forall \bar{x}^c_j$.

The interpretation of any appearance Oc of any closed formula $\forall \bar{x}^c_j A(\bar{x}^c_j)$ at a r.s.t. member Q shall consist in its 'provability' in the context of that appearance Oc conceived as the fulfillability of a deductoidal r.s.t., R, yielding $\forall \bar{x}^c_j A(\bar{x}^c_j)$ as its root formula, the r.s.t. hypotheses being only those which are accepted in the context Q of Oc. Arrows, if any, shall be counted as belonging to these hypotheses; the rule $(\widetilde{MP})$ shall be considered as applicable in R only to the arrows from Q. But now I drop arrows from attention because in the context of (Ext) no arrow precedes the formula. That is an instance of the general thesis of the equivalence of truth and provability

of a sentence, both terms to be understood relative the context where the sentence occurs. (In the Reasoning Theory, I consider justifications of logical postulates in accordance with a general form of this thesis.)

So, I shall now interpret the closed $\forall \bar{x}^b B$, $\forall \bar{x}^c B$ in (Ext) as equivalent, each, to its 'provability' in the context of Q which contains the axioms ($Dis^{bc}$). By means of a dummy Ca each axiom ($Dis^{bc}$) can be detached from the main member so that Q will not contain an arrow; the "deductoidal r.s.t." R for $\forall \bar{x}^i B$ shall be, by this definition, specified as one formalized by means of only such postulates which are justified (or "half-proved") already - so that (Ext) cannot be used in the r.s.t. as an axiom, and an axiom ($Dis^{bc}$) with $b < c$ cannot be used either (unless $x^b$, $x^c$ don't occur in the given r.s.t. variable arrows.) The definition can contain further specifications on the r.s.t. Viz. I shall assume that the last step of the r.s.t. R is a $\underset{\sim}{Ca}$ yielding that conclusion - this time, by the said reason, without any arrows.

Now, assume the antecedent $\forall \bar{x}^b B$ of the hypothesis (Ext) of the r.s.t. This assumption yields that $\forall \bar{x}^b B$ is 'provable', $\vdash \forall \bar{x}^b B$, by a r.s.t. as I have specified that. Further, it suffices that the $\underset{\sim}{Ca}$ shall be additionally specified as belonging to a kind, $K_{\underset{\sim}{Ca}}$, of $\underset{\sim}{Ca}$'s such that: (a) any $\underset{\sim}{Ca}$ of a shape actually used in the given r.s.t. belongs to $K_{\underset{\sim}{Ca}}$ and (b) any $\underset{\sim}{Ca}$ to be used for the justification of the logical axioms of the r.s.t. also belongs to $K_{\underset{\sim}{Ca}}$.

All these Ca's possess the property of <u>extendability</u> - the "<u>ext</u>-property" - consisting in that their induction variable $\bar{x}^b_j$ can be replaced by $\bar{x}^c_j$ ($b < c \leq w_j$) throughout the conclusion sentence and also in arrows, if any. The 'swallowed' arrow and those 'covered' by them continue to hold under this replacement. The rest of the arrows, unless 'absorbed', survive until the step $\overrightarrow{MP}$. The restriction on $\overrightarrow{MP}$ in R (see above) prevents the occurrence of these arrows in the r.s.t. R of the interpretations of the antecedent of (Ext) or (Res). That means, of course, that the class of the $\underset{\sim}{x}_j$-premises shall be respectively extended. (The $\underset{\sim}{Ca}$'s possess also the (inverse) property of <u>restrictability</u> - the "<u>res</u>-property" - to be used for the half-justification of the axioms (Res).) Therefore, the assumption $\forall \bar{x}^b B$ also yields $\vdash \forall \bar{x}^c B$ - and, by the tautological nature of the "proofs", $\vdash \forall \bar{x}^c B$ yields $\forall \bar{x}^c B$. Thus, $\forall \bar{x}^b B$ yields $\forall \bar{x}^c B$, which is the content of the hypotheses (Ext).

Notice that (Ext) has to be justified only in its uses for the derivation of ($Dis^{bc}$), and ($Dis^{bc}$) has to start with non-void $\forall$. in order for the problem to arise. (b, c are to be the numbers of the free variables in $\forall \bar{x}B$, $\forall \bar{x}C$ as considered up to section 14.) The same $\forall$. has, in each case, the instance of (Ext) in the "main" r.s.t.

member. Unless all quantifiers in this $\forall$. are fictitious w.r.t. $\forall \bar{x}_j^b B$, no further question arises about the applicability of that justification of $(Dis^{bc})$ in the "main" member. If $\forall$. consist in (Ext) only of fictitious quantifiers, then the following device suffices for a removal of the problems concerning the use of arrows in the r.s.t. member Q when the instance of $(Dis^{bc})$ occurs (Q may but need not be the 'main' member): choose a variable $\bar{u}$ which does not occur in Q and replace the logical axiom $(Dis^{bc})$ in Q by the formula $\forall \bar{u}(\bar{u} = \bar{u}\ \&\ (Dis^{bc}))$ obtained from $(Dis^{bc})$ by the Ca with the premises $\underline{u} = \underline{u}\ \&\ (Dis^{bc})$. Now, by means of the SBA with the arrow $\bar{u} \rightarrow \underline{a}$ and MP, the formula $\underline{a} = \underline{a}\ \&\ (Dis^{bc})$, and then $(Dis^{bc})$ is obtained, and this occurrence of $(Dis^{bc})$ shall be used in the (modified) Q instead of the former. This $\underline{a}$ shall be chosen of the same kind as $\bar{u}$ and such that $f(\underline{a}) = 0$. The arrows $\bar{u} \rightarrow \underline{a}$ shall survive up to the step of $\overrightarrow{MP}$ in the 'main' member, but that does not create any problem. The achievement of this procedure is simply that the axioms (Ext), (Res) used for $(Dis^{bc})$ no longer occur as axioms in the modified Q; they occur only in a r.s.t. member which does not contain arrows.

That shall become, upon appropriate specifications, a method of justification of the axioms (Ext). {Footnote 22}. This method has to be compatible with the interpretations of $\forall \bar{x}$'s in other logical postulates, including the rule (Ca).

Just for this purpose the item b) was included in the specification of the class $K_{\underset{\sim}{Ca}}$ above. The justifications have been considered for all logical axioms and their uses in the context of the r.s.t.'s at issue. However, they must still be considered with caution because the work was done in many stages; the assignments of ranges described above were found in 1979-80, after the Reasoning Theory was displayed - for purposes still connected with the use of different Nn's - in early 1977. Now, the whole work is to be put together in order to certify that the justifications are compatible with each other.

Too many technicalities are invloved to present a definite answer today. The justification of (Ext) - if acceptable in the context of the paradox mentioned on page 19 - shall also yield a finitized case of that paradox. Even though the explanation of [4, part VI] still applies, the question still arises about the whole work, viz.: is it a finitization of the theories dealing with infinity - or, is it rather an infinitization of a supposedly finite field?

I recall that - in accordance with the just mentioned explanations of the paradox - it arises because of the involvement of the identifications of $a^0_{\nu^0(\underline{\alpha}_h)}$ with $a_h$ for instances of $h$ not restricted to

satisfy $h < \underline{m}_{-1}$ (In the case of the paradox, the catching [3, p. 22] occurs; the use of $a^0_h$ instead of $a^0_{\nu\varphi(\underline{\alpha}_h)}$ entitles one to overcome the restrictive conditions on the values of $\underline{\alpha}_h$ which may occur in the "main" r.s.t. member with the given bound $\underline{m}_{-1}$ on the events of $D^{-1}$ denoted by the constant termoids in the "main" member of the r.s.t. The natural demand consists in that such identifications shall be performed explicitly, or by explicitly stated rules.)

2. Notice that outside the r.s.t. members the systems $\underline{S}'$, $\underline{S}^*$ with the axioms Ext, Res - or $\text{Dis}^{bc}$ with any $b, c \leq \underline{w}_j$ for the variables of the kind j - are not to be "consistent with the model". The numbers $\underline{m}_j$, $\underline{w}_j$ determine a "border of the satisfaction" for these axioms by the model. Say,

$$\bar{y}^{\underline{w}_j}_j \rightarrow \phi(\bar{x}^{\underline{w}_j-1}_j) \vdash \forall \bar{x}^{\underline{w}_j-1}_j \exists \bar{y}^{\underline{w}_j}_j (\bar{y}^{\underline{w}_j}_j = \phi(\bar{x}^{\underline{w}_j-1}_j))$$

together with the Ext

$$\forall \bar{x}^{\underline{w}_j-1}_j \exists \bar{y}^{\underline{w}_j}_j (\bar{y}^{\underline{w}_j}_j = \phi(\bar{x}^{\underline{w}_j-1}_j)) \supset \forall \bar{x}^{\underline{w}_j}_j \exists \bar{y}^{\underline{w}_j}_j (\bar{y}^{\underline{w}_j}_j = \phi(\bar{x}^{\underline{w}_j}_j))$$

yield by **MP** the false sentence

$$\bar{y}^{\underline{w}_j}_j \rightarrow \phi(\bar{x}^{\underline{w}_j-1}_j) \vdash \forall \bar{x}^{\underline{w}_j}_j \exists \bar{y}^{\underline{w}_j}_j (\bar{y}^{\underline{w}_j}_j = \phi(\bar{x}^{\underline{w}_j}_j)).$$

But this sentence does not occur in the r.s.t. at issue (because in S', S* these $\bar{x}_j$, $\bar{y}_j$ in the last formulas should have different depths).

Moreover - there is the possibility to specify $\underline{m}_j$, $\underline{w}_j$ so that, besides Res and Ext also the implications

SP1 $\qquad \forall . F \supset F^*$

SP2 $\qquad \forall . F^* \supset F$

shall hold where $F^*$ is obtained from F by raising all weight superscripts by 1 - as well as the more general ones

$\text{Sp1}^d$ $\qquad \forall . F \supset F^{*d}$

$\text{Sp2}^d$ $\qquad \forall . F \supset F_{*d}$

where $F^{*d}$ $(F_{*d})$ is obtained from F by raising (lowering) all these

superscripts by just d units.

Indeed, all axioms of $\underline{S}'$, $\underline{S}^*$ keep their force if all their weight superscripts are raised (or lowered) by $d > 0$ units, provided that the numbers $\underline{m}_j$, $\underline{w}_j$ and - if $j < k$ - $\underline{z}_{j+1}$ are respectively increased, if needed. This increasing shall be actually only a specification of the choice of this number in the definition of the model; let some numbers $\underline{m}_j$, $\underline{w}_j$, $\underline{z}_{j+1}$ used for a model for a r.s.t. with the B.S.R. or EDP be fixed, and the new numbers $\underline{m}^*_j$, $\underline{w}^*_j$, $\underline{z}^*_{j+1}$ be chosen so as to make the justifications sketched above for Ext, Res be applicable to the formulas $\underline{F} \supset \underline{F}^{*d}$, $\underline{F} \supset \underline{F}_{*d}$ where $\underline{F}$ is obtained from F of $Sp1^d$, $Sp2^d$ by substitutions of parameters for the free variables, and $\underline{F}^{*d}$, $\underline{F}_{*d}$ are obtained by the same substitutions in $F^{*d}$, $F_{*d}$, respectively. There can be a need, in the study of the r.s.t.'s, to apply $Sp1^d$, $Sp2^d$ with $d \leq \underline{d}$ where $\underline{d}$ is a fixed number $> 0$. This number $\underline{d}$ can be described as the maximal $c - b$ or $b - c$ in the $Dis^{bc}$ to be justified times the maximal height of the construction tree of a formula in a r.s.t. member as presented in $\underline{S}^*$. (The axioms $Sp1^d$, $Sp2^d$ shall be of relevance only when applied, consecutively, to the parts F of $\underline{S}^*$-formulas in the r.s.t. members; the purposes will be made clear in 3. below.) The choice of $\underline{m}_j + \underline{d}$ as $\underline{m}^*_j$ makes the definition of the ranges still applicable, even if the weights become negative as a result of the lowering so that, from henceforth, the constant arrows of the "main" r.s.t. member continue to be correct (because the text of the "main" member is not changed and $\underline{m}^*_j - d \geq \underline{m}^*_j - \underline{d} = \underline{m}_j$). $\underline{w}^*_j$ shall be chosen as $\underline{w}_j + 2\underline{d}$; this number is $\geq$ than any superscript which occurs as a result of the "raising" in a formula of a member of the r.s.t. The correctness of all arrows is kept because the raisings shall be applied uniformly to all variables which occur in a sentence. In order to keep also the applicability of the SCa's used for the "infinity axioms" (p. 16 ), the numbers $\underline{z}_{j+1}$ $j = -1, 0, \ldots, k-1$, shall be redefined too. Namely, the definition of $\underline{z}_{j+1}$ can be expressed in terms of $\underline{m}_j$, $\underline{w}_j$ because these numbers determine the set of the events of $D^j$ to be considered in the model (cf.pp.14 ); it is easy to present $\underline{z}_{j+1}$ as $\sigma(\underline{m}_j, \underline{w}_j)$ where $\sigma$ stands for a binary primitive recursive function, and now it suffices to put $\underline{z}^*_{j+1} = \sigma(\underline{m}^*_j, \underline{w}^*_j)$, $j = -1, 0, \ldots, k-1$.

But the use of Ext for the purpose of justification of $Dis^{bc}$, $b < c$, seems not to require this alteration of the numbers $\underline{m}_j$, $\underline{w}_j$, $\underline{z}_{j+1}$, so far as only the r.s.t.'s at issue in this work are concerned.

3. The danger of paradoxes arises because of the "dangerous indentifications" of the objects $a^0_{\nu o(\underline{\alpha}_h)}$ with $a^0_h$. So - let us drop these

objects altogether. No "individuals" $\underset{\sim}{\alpha}_h$ , no "raising function" $a^0_{\nu^0(\underline{\alpha})}$ on $D^{-1}$, no "dangerous identification" - and let $k = 0$. The model deals now only with $D^0$ and the $\varepsilon$-relation on its events. The kind subscripts can be dropped, the logic is again 1-sorted. The extensionallity axiom

$$\forall x \forall y [ \forall z(z \varepsilon x \sim z \varepsilon y) \supset \forall v(x \varepsilon v \supset y \varepsilon v)]$$

is fulfilled in the model and gets a demonstroid (by which the axiom is obtained as preceeded by the arrows $z \rightarrow x$, $z \rightarrow y$.) The justification of (Ext) still works.

In this way the model becomes just a banal model for the cumulative type theory; more specifically, the language of the formal system admits formulas having parts $x^c \varepsilon y^b$ with $b \leq c$ but the instances $\underline{x}^c \varepsilon \underline{y}^b$ are false. This sytem, $T^{cu}$, clearly contains the simple type theory, $\hat{T}$, without infinity axiom, as its subsystem.

E. Specker has proved [7] that Quine's system NF [9] is equiconsistent with the extension of $\hat{T}$ obtained by postulating all closed formulas $F \sim F^*$ where $F^*$ is obtained from $F$ by raising by 1 all type subscripts. Let $\hat{T}^{Sp}$ denote this extension.

I use the type subscripts in $\hat{T}$ and the type superscripts in $T^{cu}$. Just the weight superscripts can be used as these type superscripts (the function $f$ being, from the outset, the degree or "type", p. 13 , and the use of $\leq$ in the description of the ranges being a characteristic of the "cumulative" nature of this theory. There can be divergences between the types and weights caused by the summand $\underline{m}_0$ in the definition of the ranges - but here, as in [7] , such a fixed summand can be used without any deep further discussion (cf. pp 33-34 ).

The system $T^{cu}$ can be extended by postulating all of its closed formulas $F \sim F^*$ (it suffices to deal with only $F \supset F^*$) where, again, $F^*$ is obtained from $F$ by raising the type superscripts by 1. Let this extended system be denoted by $T^{cu}_{Sp}$ . The system $\hat{T}$ can easily be imbedded in $T^{cu}$, the formulas $\forall \bar{x}_i F(\bar{x}_i)$ of $\hat{T}$ being translated by $\forall \bar{x}^i [(\forall \bar{x}^{i-1} \neg \bar{x}^i = \bar{x}^{i-1}) \supset F(\bar{x}^i)]$ . (If $\underline{m}_0 > 0$, then the variables $\bar{x}^{i-1}$ are available in the model; but also in the case of $\underline{m}_0 = 0$ this translation still works if $i-1$ is replaced by $i \dot{-} 1$ where $i \dot{-} 1 = i-1$ for $i > 0$ and $0 \dot{-} 1 = 0$.)

The axioms of $\hat{T}^{Sp}$ have translations provable in $T^{cu}_{Sp}$ . For the axioms of $\hat{T}$ that is a well-known statement. Further, the axioms $F \supset F^*$ of $\hat{T}^{Sp}$ have in $T^{cu}_{Sp}$ again the translations of the shape $F \supset F^*$ and these are provable in the model by means of the axioms (Ext), $\forall \bar{x}^b (B\bar{x}^b) \supset \forall \bar{x}^{b+1} B(\bar{x}^{b+1})$, applied consecutively from within to out-

side, with the aid of the instances of (Dis).

Thus, $T^{cu}_{Sp}$, and then $\hat{T}^{Sp}$ and then NF, can be semi-proved consistent {Footnote 20}.

These consistency semi-proofs are more removed from the paradoxes than those for $ZF_k$. However, by the earlier work by Specker [8], the infinity axiom is formally provable in NF which seems to bring a paradoxical situation in a quite finite domain again.

Anyhow - the time of traditional tranquility is past. The paradoxes, unless carefully ruled out, call for a fundamental revision of the traditional concepts and appreciations. And even independently of the paradoxes - let these traditional inclinations be attacked, just for the sake of reestablishing the priority of deep questions over the common sense answers.

## APPENDIX

In [4, part IV, esp. pp. 194-237] the demonstroids have been presented, for the non-logical axioms of $\widetilde{ZF}^i_k$, in a form slightly different from that of r.s.t.'s of the present work {Footnote 21}. Arrows have been written separately from formulas but that can be amended in a mechanical way. The covering rules were not introduced in [4], and a function symbol $\zeta(\underline{x}_j)$ with the values (if any) among the members of $\underline{x}_j$ in $D^j$ was used instead. That is a many-valued function symbol defined only for $\underline{x}_j$ with $f(\underline{x}_j) > 0$; and it is used in the absorbing arrows. Now these complications can be avoided as follows: in the demonstroids of pp. 203-215 in [4], cover the arrows $t_j \to \underline{u}_j$, $t_j \to \zeta([\underline{u}_j])$ and $t_j \to \zeta(\underline{z}_j \cup [\underline{u}_j])$, by $t_j \to [\underline{u}_j]$ and $t_j \to \underline{z}_j \cup [\underline{u}_j]$, respectively; these shall be absorbed in the rCa by $t_j \to z_j$. (There is now no need for the "auxiliary" demonstroids and the "identificational arrows"; these are gone when the rules for arrows were simplified - see [11].) In the demonstroids for the Separation axioms, pp. 215-216, the same purpose of banning the symbol $\zeta$ is achieved by the insertion of $f(y_j) \leq f(x_j)$ & ... in the scope of $\exists y_j$ from which the "induction formula" of the rCa starts; in that way, the arrows $y_j \to \underline{y}_j \cup \mathfrak{z}_j$, $y_j \to \underline{y}_j$ (which have been "absorbed" with the aid of $\zeta$) occur only in the contexts where $f(\underline{y}_j) \leq f(\underline{x}_j)$ and $f(\underline{y}_j \cup \mathfrak{z}_j) \leq f(\underline{x}_j \cup \mathfrak{z}_j)$ are available, so that the arrows can be covered by $y_j \to x_j$. In the demonstroids for the Replacement axioms, the arrow $y_j \to \underline{r}_j \cup \underline{v}_j$ (in D4b 42, p. 232) can be covered by $y_j \to \underline{r}_j$, $y_j \to \underline{v}_j$ (and then both of these can be swallowed if the same $y_j$ is chosen as the "induction variables" $r_j$, $v_j$ of the Ca's yielding D4b. 44, D4b. 50, pp. 233, 234). Besides

that, in [4] the "trivial" arrows $\overline{x} \to \overline{x}$ were preserved among the "absorbing" arrows; the notion of a "swallowed" arrow $\overline{x} \to \underline{x}$ was then unknown. That created some additional complications, now removed. Notice, however, that the "trivial" arrows actually can be just disregarded when the C-cycles and the normal assignments are at issue (because the w.a.c. b) holds for them- cf. the remark of pp. 26-27 ).

These remarks suffice for justification of all claims in the present paper concerning these demonstroids. The revised form of them is going to be prepared for publication in a forthcoming work.

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FOOTNOTES

{1} I use this word without quotation marks only to denote a "codification" of an activity - say, of that of correct reasonings - by setting out a system of rules (permissions and obligations, including prohibitions). For an ideal language its grammar has to be such a system.

{2} One does not curtail a creative mathematical line of thought because its field cannot be exhausted. In the same way, creative critical thought should not be curtailed even if new questions continue to arise indefinitely.

{3} Actually, "infinite" here is misleading; I wish to stress that the rule ($\underset{\sim}{C}a$) can be formulated independently of the infinity of the range of the induction variable.

{4} These are terms in the traditional sense, while the word "term" is reserved for interpreted termoids. I wish to stress that Hilbert's notion of a term is just that of termoid because precisely the senses (values) are ignored.

{5} The "Weak" Bernays axioms being $\forall.\ \forall\bar{x}B(\bar{x}) \supset B(\bar{x})$, $\forall.\ B(\bar{x}) \supset \exists\bar{x}B(\bar{x})$; the "trivial" $\bar{x} \to \bar{x}$ shall be (justifiedly) dropped. A Weak B.A. shall be called "Very Weak" if $\forall.$ has $\forall\bar{x}$ as its rightmost quantifier.

{6} For the sake of brevity, I drop parentheses used in the notations of the rules and use the rest of the notations to denote the applications of the rules. So, $\breve{C}a$ stands for an application of ($\underset{\sim}{C}a$), MP- for an application of (MP), etc.

{7} A serious - solved - task of this program consisted in the preparations of the identifications of variables in A's of MP's as "rewritten" from the arrows in $\Gamma$ and $\Delta$ when these variables of A occur in $\Gamma$ and also in $\Delta$. Such preparations belong to the justifications of the identifications used at the logical steps and are necessary for a rigorous treatment of the r.s.t.'s. Here I can only hint at this topic.

{8} This formulation means only that for each of these functions, $\phi(\bar{x})$, and each event $\underline{x}$ from the range of $\bar{x}$ such that $\underline{x}$ is denoted by a constant termoid in R, $\phi(\underline{x})$ shall belong to the range, whereas, say, $\phi(\phi(\underline{x}))$ is not supposed to belong to it. - But, actually, some less "obvious" ways of assignments of ranges to variables better correspond to the needs of the present work - and the idea which I have just expounded will not be followed explicitly.

{9} The "length" of a proof can be measured differently; in particular - by the maximal "complexity" - or "degree", see p.13 - of a termoid in a compositional arrow.

{10} The relations $\subset$, $\subseteq$ surely involve the quantifiers $\forall t_h$, $h = -1, 0, \ldots, k$, but in the axioms it suffices to use only $\forall t_j$.

{11} These ranges are non-void. For all matters of this paper any variable $\bar{x}_j$ has a value $\underline{a}$ with $f(\underline{a}) = 0$, because a zero of $D^j$ can be fixed as this $\underline{a}$. The arrows $\bar{x}_j \to a$ can be justifiably disregarded. But in a broader discussion variables may be allowed to have void ranges, and then the axioms (Dis), p.20 below, shall be preceded by these arrows in cases where $\bar{x}$ occurs freely in B but not in C.

{12} Here I refrain from discussing some relevant properties of collations of constant termoids. They are important but not for the assignment of finite ranges.

{13} Here I don't discuss compositions of variable arrows (only) because in cases where there are no dangerous C-cycles their correctness is entailed by the w.a.c. The rules considered here apply only to the text of a deductoid (but not to r.s.t. members).

{14} i.e. "m is the Gödel number of a proof of the formula $A_n$ with the number n."

{15} Therefore, there is an example of a formula $A \,\&\, \neg A^c$ formally provable in $\tilde{Z}F_k^{wdc}$ where $A^c$ is congruent to A.

{16} The predicate calculus with BSR seems to me to be similar to - and perhaps the same as - a form of this calculus which was studied by some author in the early '60's (in connection with a syntactical form of Relevancy Theory). Unfortunately I was unable to locate those authors' names.

{17} The fictitious quantifiers shall actually be "dropped" under the interpretation of the formulas (of De or a r.s.t. member).

{18} In such a way that the definitions of the range, (p. 14 ), and of the correctness of arrows (p. 15 ), now apply with $w\bar{x}_j^c = c$; that yields $rg\bar{x}_j^c = \{\underline{x}_j | f(\underline{x}_j) \leq \underline{m}_j + c\}$, for any occurrence of $\bar{x}_j^c$ in a r.s.t. member.

{19} This formulation assumes, of course, that the r.s.t. shall be considered in the part of $\underline{S}$* without the axioms $(Dis^{bc})$, $b \neq c$ or Ext, Res (though Res or $(Dis^{bc})$ with $c \leq b$ might be accepted); these axioms still can be used in the r.s.t., but only as the hypotheses. Other distributivity axioms found redundant given these $(Dis^{bc})$ (see p. 27 ) occur in the members of the r.s.t. only as dependent on these hypotheses.

{20} In that way NF can be semi-justified without the weakening of its extensionality axiom, as in [10].

{21} The references to [4] in this work include also the references to the review of [4] - "Referativny Journal, 1970, N6, P. 10, VINITI, Moscow, USSR - where some amendments have been indicated. Two further amendments; the numbers $\underline{z}_\ell^j$ , j = 1,..., k on p. 89 of [4] - which are the $\underline{z}^j$ in the present notations - have been confused with $\check{\underline{z}}_\ell^j$ which I denote here by $\dot{\underline{z}}^j$. Correct on pp. 88-89 of [4] in accordance with p. 11 of the present paper. Reformulate the Replacement axioms as on p. 15 of the present paper.)

{22} By one of these specifications, occurrences of closed implications $A \supset B$ at a member Q of the r.s.t. shall be interpreted as $\vdash A . \supset \vdash B$ where $\vdash$ stands for the 'provability' understood as on p.30 . Open parts of formulas of Q occur only as parts of the closed ones and shall be interpreted as the interpretations of the binding quantifiers develop.

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